Telescopic Series

and the Basic Properties of Infinite Series

* So far we talked about Geometric Series

$$\left(\sum_{n=0}^{\infty} ar^n \begin{array}{l} \to \text{converges if } |r|<1 \text{ its sum } \frac{a}{1-r} \\ \to \text{diverges } a \neq 0 \text{ and } |r| \geq 1 \end{array} \right)$$

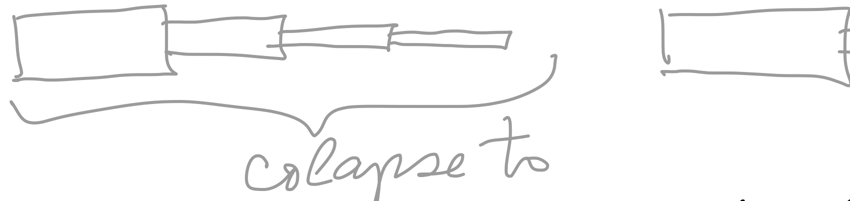
* Harmonic Series: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Harmonic numbers: $H_n = \sum_{k=0}^{n} \frac{1}{k}$, we proved

$\forall m \in \mathbb{N} \quad H_{2^m} \geq \frac{m+2}{2}$. $\left\{ \begin{array}{l} \text{The sequence of} \\ \text{Harmonic numbers} \\ \text{is unbounded.} \end{array} \right.$

* Telescopic Series (This is more like a method that comes useful in many problems.)

Telescoping



Now I will use this telescoping idea to prove that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

If $x > 1$, then $\frac{1}{(x-1)x} = \frac{1}{x-1} - \frac{1}{x} \left(= \frac{x-(x-1)}{(x-1)x} \right)$

(you might have seen this in Math 125 as a method to find indefinite integrals (partial fractions))

Let $k > 1$. $\frac{1}{k^2} \underset{\text{Pizza-Party}}{\le} \frac{1}{(k-1)k} = \frac{1}{k-1} - \frac{1}{k}$

Let $n > 1$ and calculate

many Pizza-Parties

$$S_n = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n-1)^2} + \frac{1}{n^2} \leq$$

$$\leq \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-2)(n-1)} + \frac{1}{(n-1)n}$$

$$= \frac{1}{1} + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

Telescoping

$$= 1 + 1 - \frac{1}{n} \leq 2 \quad \forall n > 1$$

We proved that

$\forall n \in \mathbb{N} \quad S_n = \sum_{k=1}^{n} \frac{1}{k^2} \leq 2$

Clearly $\{S_n\}$ is an increasing sequence

since $S_{n+1} - S_n = \frac{1}{(n+1)^2} > 0$.

By MCT the sequence $\{S_n\}$ converges

Thus the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ Converges

We used a similar trick to prove that $\sum_{k=0} \frac{1}{k!}$ converges

What is the sum of this series?

Euler named finding the sum of this series Basel Problem

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}}$$

I hope that after years of trying I wrote a proof that YOU can understand

Please google Basel Problem

Contrast to $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ DIVERGES

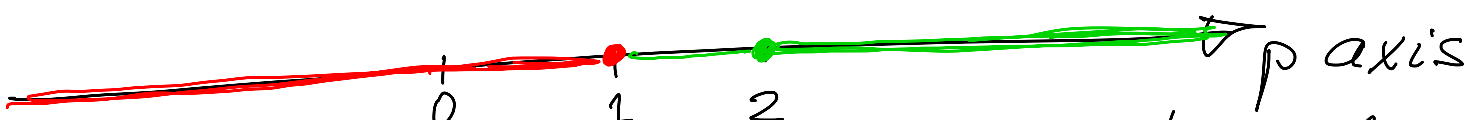
In general we might want to know for which $p \in \mathbb{R}$ the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

CONVERGES

So far we know $p=1$ Diverges

$p=2$ Converges.



• diverges

• converges

Some of this can be explained by the fact that $\frac{1}{n^p}$ is decreasing fun of p

$$\frac{1}{n} > \frac{1}{n^2}$$

This property of an infinite series is called the DIVERGENCE TEST

<u>Theorem</u> Let $\sum a_n$ be an infinite series. If $\sum_{n=1}^{\infty} a_n$ converges, then

$$\lim_{n\to\infty} a_n = 0.$$

the contrapositive of the preceding implication is "more" useful:

If the sequence $\{a_n\}$ does not converge to 0, then the series $\sum a_n$ diverges.

<u>Proof.</u> Assume $\sum_n a_n$ converges.

Set $\forall n \in \mathbb{N}$ $S_n = \sum_{k=1}^{n} a_k$. Since $\sum a_n$ converges, $\lim_{n \to +\infty} S_n = L$ for some $L \in \mathbb{R}$.

We can prove as an exercise that

$\lim_{n \to +\infty} S_{n-1} = L$. Recall

$S_n - S_{n-1} = a_n$

$\forall n \in \mathbb{N}$

Use the algebra of limits to calculate

$$\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} (S_n - S_{n-1}) = L - L = 0$$

(I should prove this with $\varepsilon - N(\varepsilon)$ from the definition ?)

Exercise 3.24 (d) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{n+3}}{\pi^{n-1}}$

This might be a Geometric Series!
How do I check that

$ar^{n-1}, ar^n, \underbrace{ar^{n+1}}, \ldots$

$\text{next}/\text{previous} = \text{constant } r$

$\dfrac{e^{n+1}}{\pi^{n-1}}, \dfrac{e^{n+2}}{\pi^{n}}, \quad \dfrac{\text{next}}{\text{previous}} \quad \underbrace{\dfrac{\dfrac{e^{n+2}}{\pi^{n}}}{\dfrac{e^{n+1}}{\pi^{n-1}}}}_{\text{simplify}} =$

next

$= \dfrac{e}{\pi}$ (this is r, since $\frac{e}{\pi} < 1$ this series converges.)