# Alternating Series

## Conditional Convergence

The most famous alternating series is

$$\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

this is Alternating Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$ (diverges)

Harmonic Series

Amazingly, or maybe not so amazingly the Alternating Harmonic Series Converges. Here is Why:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

To verify convergence we check the PARTIAL SUMS of a series

$$S_1 = 1$$

$$S_2 = \left(1 - \frac{1}{2}\right)$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{3}$$

$$S_4 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$$

For all even partial sums:

$$\frac{1}{2} + \frac{1}{12} + \dots$$

| | A | |
|---|---|---|
| | terms | partial sums |
| 1 | 1 | 1 |
| 2 | $-1/2$ | = sum( ) |
| 3 | $1/3$ | = sum(A$1:A3) |
| | $-1/4$ | = sum(A$1:A4 |
| | $1/5$ | |

$$S_{2n} = \left(1-\frac{1}{2}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right)$$

$$= \frac{1}{2} + \frac{1}{12} + \cdots + \underbrace{\frac{2n-(2n-1)}{2n(2n-1)}}_{= \frac{1}{2n(2n-1)}}$$

We should be able to come up with a bound $S_{2n} < 1$ (prove it?)

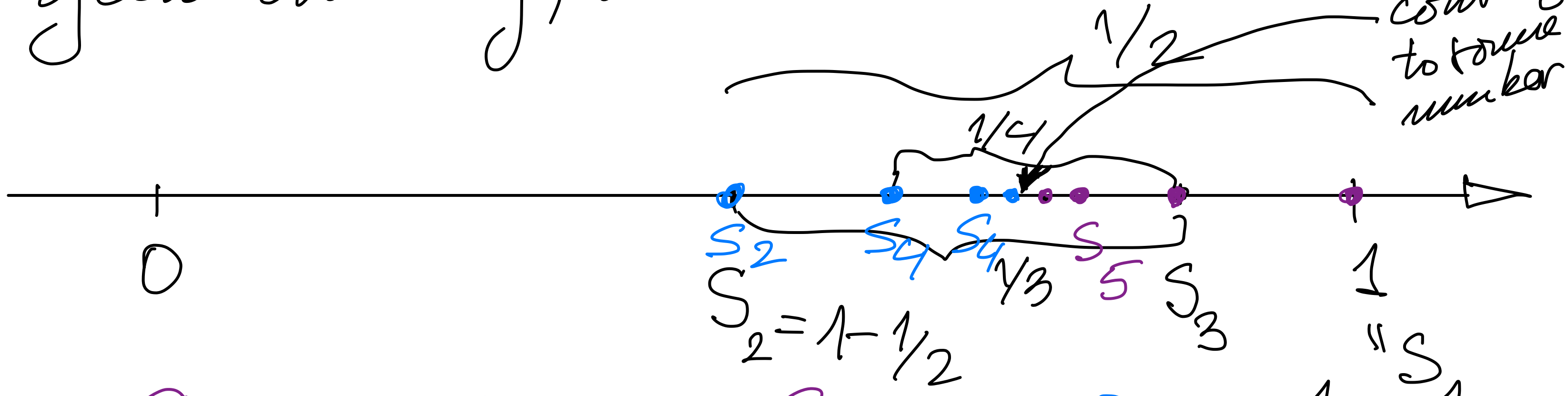
We can see that $S_{2n} < S_{2(n+1)}$

bdd

Increas.

So MCT $\Rightarrow$ $S_{2n}$ $\Rightarrow$ Converges

Look at the PARTIAL SUMS geometrically, on a number line




$$S_{2n-1} - S_{2m} \le S_{2m-1} - S_{2m} = \frac{1}{2m}$$

small
large m

blue dots are increasing

$n \ge m$ $\Rightarrow$ $S_{2m-1} \ge S_{2n-1}$ $\quad S_{2n} \ge S_2$

$$S_{2m-1} - S_{2m} = \boxed{\sum_{k=1}^{2m-1} \frac{(-1)^{k+1}}{k}} - \boxed{\sum_{k=1}^{2m} \frac{(-1)^{k+1}}{k}}$$

$$= - \frac{(-1)^{2m+1}}{2m} = \frac{1}{2m}$$

Remark: In the notes, I proved that

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln 2 \quad \left\{ \text{using the Riemann sums } \int_1^2 \frac{1}{t}\,dt \right)$$

Instead of $\frac{1}{n}$ in the AHS, we could use any sequence $a_n$ s.t. $a_n > 0$ $a_n \geq a_{n+1}$ and $\lim_{n\to\infty} a_n = 0$.

## Alternating Series Test:

Assume

1. $a_n > 0$ $\forall n \in \mathbb{N}$
2. $a_n \geq a_{n+1}$ $\forall n \in \mathbb{N}$
3. $\lim_{n\to\infty} a_n = 0$

then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ CONVERGES

Now comes an amazing fact about Alternating Harmonic Series.

You can think of an infinite series as balancing an infinite checkbook

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots$$

| $1$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ | $\frac{1}{6}$ | $\frac{1}{7}$ |
|---|---|---|---|---|---|---|
| $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
| d | w | d | w | d | w | d |

Total deposits | total withdr.

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots \qquad \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots$$

$$>\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots \qquad \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)$$

$$>\frac{1}{2}\,\text{Har} \qquad \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$$

~~Conv.~~

$\boxed{\text{diver.}}$

$$\sum_{n=0}^{\infty}\frac{1}{n!}$$

Both Series Diverge

Sloppy way we can say:

In this account $\infty$\$ is dep. \$ $\infty$ is withdrawn

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}\dots$$

In the notes the sum changes to

$$\frac{1}{2}\ln 2$$

Therefore Convergent Series with "Infinite" amount of deposits and "infinite" amo of w. are

Called

CONDITIONALLY

CONVERGENT