On minimums and maximums

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Definition 1. Let A and B be two sets. We say that A is a *subset* of B, and write $A \subseteq B$, if and only if for every $x \in A$ we have $x \in B$. In notation

$$A \subseteq B \qquad \Leftrightarrow \qquad \forall x \ \big(x \in A \Rightarrow x \in B\big), \tag{1}$$

Definition 2. Let S be a subset of $\mathbb{R}$. If u is the smallest number in S, then u is called a *minimum* of S and we write $u = \min S$. If v is the greatest number in S, then v is called a *maximum* of S and we write $v = \max S$. More formally, we express these definitions as logical statements:

$$u = \min S \qquad \Leftrightarrow \qquad (u \in S) \wedge \big(\forall x \in S \ \ u \leq x\big), \tag{2}$$

$$v = \max S \qquad \Leftrightarrow \qquad (v \in S) \wedge \big(\forall x \in S \ \ x \leq v\big). \tag{3}$$

Proposition 3. *Let A and B be nonempty subsets of $\mathbb{R}$ such that $A \subseteq B$. The following statements hold.*

(i) *If the sets A and B have minimums, then*

$$\min B \leq \min A. \tag{4}$$

(ii) *If the sets A and B have maximums, then*

$$\max A \leq \max B. \tag{5}$$

Proof. Assume A and B be nonempty subsets of $\mathbb{R}$ such that $A \subseteq B$.

(i) Assume that A and B have minimums and set $a = \min A$ and $b = \min B$. By definition of the minimum, see $\Rightarrow$ in (2), we have $a \in A$. By definition of the subset, see $\Rightarrow$ in (1), $a \in A$ implies $a \in B$. Hence $a \in B$ holds. Since $b = \min B$, by definition of the minimum, see $\Rightarrow$ in (2), we have that $b \leq y$ for all $y \in B$. Since we already proved that $a \in B$, we conclude that $b \leq a$. This proves (4).

(ii) Assume that A and B have maximums and set $c = \max A$ and $d = \max B$. By definition of the maximum, see $\Rightarrow$ in (3), we have $c \in A$. By definition of the subset, see $\Rightarrow$ in (1), we deduce that $c \in B$. Since $d = \max B$, by definition of the maximum, see $\Rightarrow$ in (3), we have that $y \leq d$ for all $y \in B$. Since we already proved that $c \in B$, we conclude that $c \leq d$. This proves (5). □

Definition 4. Let A and B be nonempty subsets of $\mathbb{R}$. We define the *sum* $A+B$ to be the following set

$$A+B = \big\{x+y \in \mathbb{R} : (x \in A) \wedge (y \in B)\big\}. \tag{6}$$

Proposition 5. *Let A and B be nonempty subsets of $\mathbb{R}$. The following statements hold.*

(i) *If the sets A and B have minimums, then*

$$\min(A+B) = \min A + \min B. \tag{7}$$

(ii) *If the sets A and B have maximums, then*

$$\max(A+B) = \max A + \max B. \tag{8}$$

Proof. Assume A and B be nonempty subsets of $\mathbb{R}$.

(i) Assume that A and B have minimums and set $a = \min A$ and $b = \min B$. By definition of the minimum, see $\Rightarrow$ in (2), we have that $a \in A$ and $b \in B$. By definition of the sum for two sets, see (6), we have $a+b \in A+B$. By definition of the minimum, see $\Rightarrow$ in (2), we have that $z \geq \min(A+B)$ for all $z \in A+B$. Since we already proved that $a+b \in A+B$, we conclude that $a+b \geq \min(A+B)$. This proves (7).

(ii) Assume that A and B have maximums and set $c = \max A$ and $d = \max B$. By definition of the maximum, see $\Rightarrow$ in (3), we have that $a \in A$ and $b \in B$. By definition of the sum for two sets, see (6), we have $a+b \in A+B$. By definition of the maximum, see $\Rightarrow$ in (3), we have that $z \leq \max(A+B)$ for all $z \in A+B$. Since we already proved that $a+b \in A+B$, we conclude that $a+b \leq \min(A+B)$. This proves (8). $\square$