The number e is irrational

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Here we use the following definition of e:

$$e = \lim_{n\to+\infty} \sum_{k=0}^{n} \frac{1}{k!}.$$

Lemma 1. *For every $m, n \in \mathbb{N}$ we have*

$$\sum_{k=0}^{n} \frac{1}{k!} \leq \frac{2}{(m+1)!} + \sum_{k=0}^{m} \frac{1}{k!}. \tag{1}$$

Proof. Let $m, n \in \mathbb{N}$. For $n \leq m$ the inequality is clear. If $n > m$ we have

$$\begin{aligned}
\sum_{k=0}^{n} \frac{1}{k!} &= \sum_{k=0}^{m} \frac{1}{k!} + \sum_{k=m+1}^{n} \frac{1}{k!} && \text{algebra} \\
&= \sum_{k=0}^{m} \frac{1}{k!} + \frac{1}{m!} \sum_{k=m+1}^{n} \frac{m!}{k!} && \text{algebra} \\
&= \sum_{k=0}^{m} \frac{1}{k!} + \frac{1}{m!} \left(\frac{1}{m+1} + \frac{1}{(m+1)(m+2)} + \cdots + \frac{1}{(m+1)\cdots n} \right) && \text{algebra} \\
\text{pizza-party} \quad &\leq \sum_{k=0}^{m} \frac{1}{k!} + \frac{1}{m!} \left(\frac{1}{m+1} + \frac{1}{(m+1)(m+2)} + \cdots + \frac{1}{(n-1)n} \right) \\
&= \sum_{k=0}^{m} \frac{1}{k!} + \frac{1}{m!} \left(\frac{1}{m+1} + \left(\frac{1}{m+1} - \cancel{\frac{1}{m+2}} \right) + \cancel{\cdots} + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n} \right) \right) && \text{algebra} \\
&= \sum_{k=0}^{m} \frac{1}{k!} + \frac{1}{m!} \left(\frac{2}{m+1} - \frac{1}{n} \right) && \text{drop } -\frac{1}{n} < 0 \\
&\leq \sum_{k=0}^{m} \frac{1}{k!} + \frac{2}{(m+1)!}
\end{aligned}$$

□

The following theorem is proved somewhere else. It is the background knowledge in this context.

Theorem 2. *Let $L \in \mathbb{R}$ and let $\{s_n\}$, be a convergent sequence with the limit L. Let $a, b \in \mathbb{R}$ be such that for some $n_0 \in \mathbb{N}$ we have*

$$a \leq s_n \leq b$$

for all $n \in \mathbb{N}$ such that $n \geq n_0$. Then $a \leq L \leq b$.

$m \in \mathbb{N}$ is fixed. Let $n \in \mathbb{N}$ be s.t.

$n \geq m$. Then $\displaystyle\sum_{k=0}^{m} \frac{1}{k!} \leq \sum_{k=0}^{n} \frac{1}{k!} \leq \frac{2}{(m+1)!} + \sum_{k=0}^{m} \frac{1}{k!}$

$n \to \infty$ $\quad \to e$

Applying Theorem 2 to inequality (1) and the definition of e we obtain the following corollary.

Corollary 3. *For every $m \in \mathbb{N}$ we have*

$$\boxed{\sum_{k=0}^{m} \frac{1}{k!} < e \leq \frac{2}{(m+1)!} + \sum_{k=0}^{m} \frac{1}{k!}.} \tag{2}$$

In particular, with $m = 3$,

$$\frac{8}{3} < e < \frac{11}{4}. \tag{3}$$

Theorem 4. *For all $p \in \mathbb{Z}$ and and all $q \in \mathbb{N}$ we have*

$$e \neq \frac{p}{q}. \tag{4}$$

Proof. Since e is positive, (4) holds for all $q \in \mathbb{N}$ and all $p \in \mathbb{Z}$ such that $p \leq 0$.

Let $q \in \mathbb{N}$ be such that $q > 1$. By (2) we have

$m = q$

$*q!$

$$0 < q! \left(e - \sum_{k=0}^{q} \frac{1}{k!} \right) \leq q! \frac{2}{(q+1)!} = \frac{2}{q+1} \leq \frac{2}{3}.$$

If $q = 1$, then by (3)

$$0 < 1! \left(e - \sum_{k=0}^{1} \frac{1}{k!} \right) = e - 2 \leq \frac{3}{4}.$$

Fourier

From the preceding two displayed inequalities we have

$$\boxed{\forall q \in \mathbb{N} \quad q! \left(e - \sum_{k=0}^{q} \frac{1}{k!} \right) \notin \mathbb{Z}.} \tag{5}$$

Let $p, q \in \mathbb{N}$. Then

$$q! \left(\frac{p}{q} - \sum_{k=0}^{q} \frac{1}{k!} \right) = p(q-1)! - \sum_{k=0}^{q} \frac{q!}{k!}. \tag{6}$$

Since

$$\forall k \in \{0, 1, \ldots, q\} \qquad \frac{q!}{k!} \in \mathbb{Z},$$

equality (6) yields

$$\boxed{\forall p \in \mathbb{N} \ \ \forall q \in \mathbb{N} \qquad q! \left(\frac{p}{q} - \sum_{k=0}^{q} \frac{1}{k!} \right) \in \mathbb{Z}} \tag{7}$$

From (5) and (7) we deduce

$$\forall p \in \mathbb{N} \ \ \forall q \in \mathbb{N} \qquad \boxed{q! \left(e - \sum_{k=0}^{q} \frac{1}{k!} \right) \neq q! \left(\frac{p}{q} - \sum_{k=0}^{q} \frac{1}{k!} \right)}$$

Consequently,

$$\forall p \in \mathbb{N} \ \ \forall q \in \mathbb{N} \qquad e \neq \frac{p}{q}.$$

Together with the first sentence of this proof, this proves the theorem. $\square$

This is MATH in my VISION of it.