All about the set $\{x \in \mathbb{Z} : x > 0 \text{ and } x \leq 1\}$

Proposition 1. The only element of the set $\{x \in \mathbb{Z} : x > 0 \text{ and } x \leq 1\}$ is the number 1.

Proof. Set $S := \{x \in \mathbb{Z} : x > 0 \text{ and } x \leq 1\}$. Since we proved $0 < 1$ and clearly $1 \leq 1$, we have $1 \in S$. Thus $S \neq \emptyset$. Since by definition $0 < x$ for all $x \in S$, the set S is bounded below by zero. Therefore, the Well-Ordering Axiom the set S has a minimum. Set $m := \min S$. The integer m has the following two properties:

$$m \in S,$$
$$m \leq x \quad \text{for all} \quad x \in S.$$

Since $m \in S$, $m > 0$ and $m \leq 1$. Therefore, by Axiom 15 applied twice, $m^2 > 0$ and $m^2 \leq m$. The last inequality, Axiom 13 and $m \leq 1$ imply $m^2 \leq 1$. Hence $m^2 \in S$. Since m is the minimum of S, we have $m \leq m^2$. Since we already proved $m^2 \leq m$, it follows that $m^2 = m$; that is $m\,m = 1\,m$. Since we know that $m \neq 0$, Axiom 10 implies $m = 1$.

Let x be an arbitrary number in S. Then, by definition of S, $x \leq 1$. Since $1 = \min S$, $1 \leq x$. Hence $x = 1$. This proves that 1 is the only element in S. □

Proposition 2. Let $k \in \mathbb{Z}$. Then $\{x \in \mathbb{Z} : x > k \text{ and } x \leq k+1\} = \{k+1\}$.

Proof. Set, as before,

$$S := \{x \in \mathbb{Z} : x > 0 \text{ and } x \leq 1\}$$

and

$$T := \{x \in \mathbb{Z} : x > k \text{ and } x \leq k+1\}.$$

We will prove the following equivalence:

$$x \in T \quad \Leftrightarrow \quad x - k \in S.$$

Assume $x \in T$. Then $x > k$ and $x \leq k + 1$. By Axiom 14, $x - k > 0$ and $x - k \leq 1$. Therefore $x - k \in S$.

Now assume $x - k \in S$. Then, $x - k > 0$ and $x - k \leq 1$. By Axiom 14, $x > k$ and $x \leq k + 1$. Therefore $x \in T$.

By Proposition 1, $x - k \in S$ if and only if $x - k = 1$. Clearly, $x - k = 1$ if and only if $x = k + 1$. Thus,

$$x - k \in S \quad \Leftrightarrow \quad x = k + 1.$$

The last two displayed equivalences yield

$$x \in T \quad \Leftrightarrow \quad x = k + 1.$$

The last equivalence implies $T = \{k + 1\}$. $\square$