Sixteen and Seventeen Consecutive Integers

Lemma 1. Let $A = \{a, a+2, a+4, a+6, a+8, a+10\}$ be a set of six consecutive odd integers. Assume that each of the integers in A is divisible by at least one of the integers 3, 5, 7 and 11. Then either $a+4$ or $a+6$ is not divisible by any of $3, 5,$ and 7.

Proof. Denote by A_3 the set of integers in A that are divisible by 3, denote by A_5 the set of integers in A that are divisible by 5, denote by A_7 the set of integers in A that are divisible by 7, and denote by A_{11} the set of integers in A that are divisible by 11. Studying three cases for a, it can be proved that A_3 has exactly 2 elements. If a is divisible by 5, then A_5 has 2 elements. If a is not divisible by 5, then A_5 has 1 element. The set A_7 has at most 1 element and A_{11} has at most 1 element. (This can be proved by studying all the possible remainders for a when divided by 7 and 11.) Thus, in order to have the equality $A_3 \cup A_5 \cup A_7 \cup A_{11} = A$ each of the sets A_3, A_5, A_7, A_{11} must have its maximum number of elements and these sets must not overlap. In order for A_5 to have two elements we must have $a \in A_5$ and consequently $A_5 = \{a, a+10\}$. In order for A_3 to have two elements without overlapping with A_5 we must have $a+2 \in A_3$ and consequently $A_3 = \{a+2, a+8\}$. The only remaining integers in A are $a+4$ and $a+6$ and one of them must be in A_{11} and the other one in A_7. The integer which is in A_{11} is not divisible by $3, 5, 7$. □

Lemma 2. Among any seven consecutive odd integers there is at least one which is not divisible by 3, 5, 7 and 11.

Proof. Among any 7 consecutive odd integers there are at most three integers divisible by 3, there are at most two integers divisible by 5, there is at most one integer divisible by 7 and there is at most one integer divisible by 11. Denote by C a set of 7 consecutive odd integers, that is $C = \{a, a+2, a+4, a+6, a+8, a+10, a+12\}$. Denote by C_3 the set of integers in C that are divisible by 3, denote by C_5 the set of integers in C that are divisible by 5, denote by C_7 the set of integers in C that are divisible by 7, and denote by C_{11} the set of integers in C that are divisible by 11. In order for the set C_3 to have three elements (the largest possible case) a must be divisible by 3. In this case $C_3 = \{a, a+6, a+12\}$. If $a+2$ is divisible by 3, we have $C_3 = \{a+2, a+8\}$ and if $a+4$ is divisible by 3, we have $C_3 = \{a+4, a+10\}$. If C_3 has only two elements, then the union $C_3 \cup C_5 \cup C_7 \cup C_{11}$ can have at most six elements. Since C has 7 elements, in this case there must be an integer in C which is NOT in $C_3 \cup C_5 \cup C_7 \cup C_{11}$. This integer is not divisible by any of 3, 5, 7 and 11.

As I mentioned above if C_3 has three elements, then a must be divisible by 3. Assume that a is divisible by three, and consider three possible cases for C_5. Note that in this case $C_3 = \{a, a+6, a+12\}$. If a is divisible by 5, then $C_5 = \{a, a+10\}$, and in this case $C_3 \cup C_5$ has four elements. If $a+2$ is divisible by 5, then $C_5 = \{a+2, a+12\}$ and again $C_3 \cup C_5$ has four elements. If the least integer in C_5 is greater than $a+2$, then C_5 has only one element, and the set $C_3 \cup C_5$ has at most four elements. Thus in any case the set $C_3 \cup C_5$ has at most four elements. Since each set C_7 and C_{11} has only one element we conclude that the set $(C_3 \cup C_5) \cup C_7 \cup C_{11}$ has at most six elements. Since C has 7 elements, again we conclude that there must be an integer in C which is NOT in $C_3 \cup C_5 \cup C_7 \cup C_{11}$. This integer is not divisible by 3,5,7 and 11. □

Lemma 3. Among any eight consecutive odd integers there is at least one which is not divisible by 3, 5, 7, 11 and 13.

Disproof of Lemma. This lemma is not true as the following example shows:

$$189623 = 27089 \cdot 7, \quad 189625 = 37925 \cdot 5, \quad 189627 = 63209 \cdot 3,$$
$$189629 = 17239 \cdot 11, \quad 189631 = 14587 \cdot 13,$$
$$189633 = 63211 \cdot 3, \quad 189635 = 37927 \cdot 5, \quad 189637 = 27091 \cdot 7$$

□

Problem 4. Prove that among any sixteen consecutive integers there is always at least one integer relatively prime to the other fifteen.

Proof. Let $S = \{b, b+1, \dots, b+14, b+15\}$ be a set of 16 consecutive integers. Assume that b is even. The case of an odd b is considered in a similar way. There are 8 consecutive odd integers in S: $\{b+1, b+3, \dots, b+13, b+15\}$. I will consider two cases:

Case 1. Among the six consecutive odd integers $\{b+1, b+3, \dots, b+11\}$ each is divisible by at least one of the integers 3, 5, 7 and 11.

Case 2. At least one of the six consecutive odd integers $\{b+1, b+3, b+5, b+7, b+9, b+11\}$ is not divisible by any of the numbers 3, 5, 7 and 11. Call this integer f. In this case I will consider two subcases

Case 2a. $f = b+1$.

Case 2b. $b+3 \leq f \leq b+11$.

Case 1. Assume that among the six consecutive odd integers $\{b+1, b+3, \dots, b+11\}$ each is divisible by at least one of the integers 3, 5, 7 and 11. Lemma 1 applies in this case. Using Lemma 1, I conclude that either $b+5$ or $b+7$ is not divisible by 3, 5 and 7. Denote this odd integer by u. Then $|u-x| \leq 10$ for all $x \in S$ and u is not divisible by any of 3, 5 and 7. The last two facts about u can be used to prove that u is relatively prime to the other fifteen integers in S.

Claim. If u is an odd integer in S such that $|u-x| \leq 10$ for all $x \in S$ and such that u is not divisible by any of 3, 5 and 7, then u is relatively prime to the other in S.

Proof of the claim. Assume that u is an odd integer in S such that $|u-x| \leq 10$ for all $x \in S$ and u is not divisible by any of 3, 5 and 7. Since $|u-x| \leq 10$, I conclude that $|u-x|$ can be any of the numbers $1, 2, \dots, 10$. Let $d = \gcd(u, x)$. Assume that $d > 1$, and let p be a prime divisor of d. Since d divides one of the numbers $1, 2, \dots, 10$, I conclude that p divides one of the numbers $1, 2, \dots, 10$. Since p is a prime, I conclude that $p \in \{2, 3, 5, 7\}$. Since $d = \gcd(u, x)$, I conclude that $p \mid u$. Since u is not divisible by any of 2, 3, 5 and 7, I conclude that $p \notin \{2, 3, 5, 7\}$. Contradiction! This proves that the assumption $d > 1$ is false. □

Case 2a. Assume that the integer $b+1$ is not divisible by any of the integers 3, 5, 7 and 11. Consider the seven consecutive odd integers $\{b+3, b+5, b+7, b+9, b+11, b+13, b+15\}$. By Lemma 2, at least one of these integers is not divisible by any of 3, 5, 7 and 11. Call this integer g. Since the integer $g-(b+1) \leq 14$ is even it is not divisible by 13. Therefore at least one of the integers $b+1$ and f is not divisible by 13. Call this integer u. Then $|u-x| \leq 15$ for all $x \in S$ and u is not divisible by any of 3, 5, 7, 11 and 13. The last two facts about u can be used to prove that u is relatively prime to the other fifteen integers in S. (The proof is very similar to the proof about u given in Case 1.)

Case 2b. Assume that at least one of the five consecutive odd integers $\{b+3, b+5, b+7, b+9, b+11\}$ is not divisible by any of the numbers 3, 5, 7 and 11. Call this integer u. Then $|u-x| \leq 12$ for all $x \in S$ and u is not divisible by any of 3, 5, 7, and 11. The last two facts about u can be used to prove that u is relatively prime to the other fifteen integers in S. (The proof is very similar to the proof about u given in Case 1.) □

Problem 5. Prove that among any seventeen consecutive integers there is always at least one integer relatively prime to the other sixteen.

Disproof of the problem. Among the following seventeen consecutive integers there is no integer which is relatively prime to the other sixteen integers. The idea here is to find six consecutive odd integers g, i, k, n, q, s which satisfy the pattern of Lemma 1, with n divisible by 11, k divisible by 7, i and q divisible by 3 and g and s divisible by 5. Also choose the preceding odd integer d to be divisible by 13. Then automatically the next preceding odd integer b will be divisible by 3. Thus we can concentrate on choosing an odd integer d to be divisible by 13, $d+2$ divisible by 5, $d+4$ divisible by 3, $d+6$ divisible by 7 and $d+8$ divisible by 11. This can be done by using the method of Section 3.5. The seventeen integers are:

$$\begin{aligned} a &= 268070, & b &= 268071, & c &= 268072, \\ d &= 268073, & f &= 268074, & g &= 268075, \\ h &= 268076, & i &= 268077, & j &= 268078, \\ k &= 268079, & m &= 268080, & n &= 268081, \\ p &= 268082, & q &= 268083, & r &= 268084, \\ s &= 268085, & t &= 268086\ . \end{aligned}$$

The following table gives the greatest common divisors of these seventeen numbers:

	a	b	c	d	f	g	h	i	j	k	m	n	p	q	r	s	t
a	na	1	2	1	2	5	2	1	2	1	10	11	2	1	2	5	2
b	1	na	1	1	3	1	1	3	1	1	3	1	1	3	1	1	3
c	2	1	na	1	2	1	4	1	2	7	8	1	2	1	4	1	14
d	1	1	1	na	1	1	1	1	1	1	1	1	1	1	1	1	13
f	2	3	2	1	na	1	2	3	2	1	6	1	2	9	2	1	6
g	5	1	1	1	1	na	1	1	1	1	5	1	1	1	1	5	1
h	2	1	4	1	2	1	na	1	2	1	4	1	2	1	4	1	2
i	1	3	1	1	3	1	1	na	1	1	3	1	1	3	1	1	3
j	2	1	2	1	2	1	2	1	na	1	2	1	2	1	2	1	2
k	1	1	7	1	1	1	1	1	1	na	1	1	1	1	1	1	7
m	10	3	8	1	6	5	4	3	2	1	na	1	2	3	4	5	6
n	11	1	1	1	1	1	1	1	1	1	1	na	1	1	1	1	1
p	2	1	2	1	2	1	2	1	2	1	2	1	na	1	2	1	2
q	1	3	1	1	9	1	1	3	1	1	3	1	1	na	1	1	3
r	2	1	4	1	2	1	4	1	2	1	4	1	2	1	na	1	2
s	5	1	1	1	1	5	1	1	1	1	5	1	1	1	1	na	1
t	2	3	14	13	6	1	2	3	2	7	6	1	2	3	2	1	na

□