

Fall 2016 Math 304 Topics for the exam

On an exam you can be asked to prove any of the theorems stated in the summary.

4.7 Change of bases (in fact: Change of coordinates).

- > Know that the matrix M with the property $[\mathbf{v}]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **change-of-coordinate matrix from \mathcal{B} to \mathcal{C}** . It is denoted $P_{\mathcal{C} \leftarrow \mathcal{B}}$ and it is calculated as

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix};$$

here $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and \mathcal{C} are two bases of an n -dimensional vector space \mathcal{V} and $[\mathbf{v}]_{\mathcal{B}} \in \mathbb{R}^n$ denotes the coordinate vector of $\mathbf{v} \in \mathcal{V}$ relative to the basis \mathcal{B} .

- > $\left(P_{\mathcal{C} \leftarrow \mathcal{B}}\right)^{-1} = P_{\mathcal{B} \leftarrow \mathcal{C}}$
- > Know that there is a special basis of \mathbb{R}^n , called the **standard basis**, which consists of the columns of the identity matrix I_n . It is denoted by \mathcal{E} . This notation comes from the fact that the vectors in this basis are commonly denoted by $\vec{e}_1, \dots, \vec{e}_n$.
- > Know that for a basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ for \mathbb{R}^n (this is a special case) we have

$$P_{\mathcal{E} \leftarrow \mathcal{B}} = [\vec{b}_1 \cdots \vec{b}_n] = P_{\mathcal{B}}$$

(the matrix $P_{\mathcal{B}}$ was introduced in Section 4.4); this basically says that $P_{\mathcal{E} \leftarrow \mathcal{B}}$ is a very friendly matrix.

- > Know $P_{\mathcal{C} \leftarrow \mathcal{B}} = \left(P_{\mathcal{C} \leftarrow \mathcal{E}}\right) \left(P_{\mathcal{E} \leftarrow \mathcal{B}}\right) = \left(P_{\mathcal{E} \leftarrow \mathcal{C}}\right)^{-1} P_{\mathcal{E} \leftarrow \mathcal{B}} = [\vec{c}_1 \cdots \vec{c}_n]^{-1} [\vec{b}_1 \cdots \vec{b}_n]$
where $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$ is another basis for \mathbb{R}^n ; this basically says that $P_{\mathcal{C} \leftarrow \mathcal{B}}$ can be written as a product of a very friendly matrix and the inverse of another very friendly matrix
- > Know Exercises 4 - 10, 13, 14.

5.1 Eigenvectors and eigenvalues.

- > Know the definition of an eigenvector and eigenvalue. It is a little tricky. Pay attention.
- > Know the definition of an eigenspace and how to find an eigenspace corresponding to a given eigenvalue.
- > Know that the eigenvalues of a triangular matrix are the entries on its main diagonal.
- > **Theorem.** Eigenvectors corresponding to distinct eigenvalues are linearly independent. (Or, in more formal mathematical language: Let A be an $n \times n$ matrix, let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \in \mathbb{R}^n$ and $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$. If $A\vec{v}_k = \lambda_k \vec{v}_k$, $\vec{v}_k \neq \vec{0}$ and $\lambda_j \neq \lambda_k$ for all $j, k = 1, 2, \dots, m$, then $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ are linearly independent.
- > Know the proof of the above theorem for $m = 2$ vectors.

5.2 The characteristic equation.

- > Know that λ is an eigenvalue of an $n \times n$ matrix A if and only if $\det(A - \lambda I) = 0$
- > Know how to calculate $\det(A - \lambda I)$ for 2×2 matrices, how to find eigenvalues and corresponding eigenvectors. Exercises 1-8, but do more and find eigenvectors as well.

5.3 Diagonalization.

- > **Theorem.** (The diagonalization theorem) An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
- > Know how to decide whether a given 2×2 and 3×3 matrix A , is diagonalizable or not; if it is diagonalizable, how to find invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

5.4 Eigenvectors and linear transformations.

- Know that the matrix M with the property $[T\mathbf{v}]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **matrix of T relative to the bases \mathcal{B} and \mathcal{C}** and it is calculated as

$$M = \begin{bmatrix} [T\mathbf{b}_1]_{\mathcal{C}} & \cdots & [T\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix},$$

where T be a linear transformation from an m -dimensional vector space \mathcal{V} to an n -dimensional vector space \mathcal{W} , $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ is a basis for \mathcal{V} and \mathcal{C} is a basis for \mathcal{W} .

- Know that the matrix M with the property $[T\mathbf{v}]_{\mathcal{B}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **matrix of T relative to the basis \mathcal{B}** , it is denoted by $[T]_{\mathcal{B}}$ and it is calculated as

$$[T]_{\mathcal{B}} = \begin{bmatrix} [T\mathbf{b}_1]_{\mathcal{B}} & \cdots & [T\mathbf{b}_n]_{\mathcal{B}} \end{bmatrix},$$

where $T : \mathcal{V} \rightarrow \mathcal{V}$ is a linear transformation on an n -dimensional vector space \mathcal{V} and $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for \mathcal{V} .

- **Theorem.** (The diagonal matrix representation) Let A, D, P be $n \times n$ matrices, where P is invertible and D is diagonal. If \mathcal{B} is the basis for \mathbb{R}^n which consists of the columns of P and if $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given by $T\vec{v} = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^n$, then $[T]_{\mathcal{B}} = D$.

5.5 Complex eigenvalues.

- Know how to calculate complex eigenvalues and corresponding eigenvectors of 2×2 and 3×3 real matrices
- Know that the most important class of 2×2 matrices with complex eigenvalues are rotation matrices; the matrix of the counterclockwise rotation about the origin by the angle θ measured in radians relative to the standard basis for \mathbb{R}^2 is

$$R_{\theta} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix};$$

the 2×2 matrix R_{θ} is called the **rotation matrix**.

- **Theorem.** (A “hiding rotation” theorem.) Let A be a real 2×2 matrix with a nonreal eigenvalue. Then there exist an invertible 2×2 matrix P , a positive scalar α and a rotation matrix R_{θ} such that $A = \alpha P R_{\theta} P^{-1}$.

6.1 Inner product, length, and orthogonality.

- Know the definition of the inner product (dot product) in \mathbb{R}^n , its basic properties and calculations involving it.
- Know the definition, the basic properties of the length in \mathbb{R}^n and calculations involving it.
- Know the definition of the distance in \mathbb{R}^n and calculations involving it.
- Know the definition of orthogonality in \mathbb{R}^n and calculations involving it.
- Know the statement and the proof of the linear algebra version of **Pythagorean theorem**.
- Know the definition and the basic properties of the orthogonal complement in \mathbb{R}^n .
- Know that for a given $m \times n$ matrix A we have $(\text{Row } A)^{\perp} = \text{Nul } A$ and $(\text{Col } A)^{\perp} = \text{Nul } A^{\top}$.
- Know the geometric interpretation of the inner product (dot product) in \mathbb{R}^2 and \mathbb{R}^3 :

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \vartheta, \tag{1}$$

where \vec{u} and \vec{v} are vectors in \mathbb{R}^2 or in \mathbb{R}^3 , ϑ is the angle at the vertex O in the triangle OAB with O being the origin, A being the endpoint of \vec{u} and B the endpoint of \vec{v} . (You should know the proof of formula (1).)

6.2 Orthogonal sets.

- > Know the definition of an orthogonal set of vectors.
- > **Theorem.** (Linear independence of orthogonal sets.) Let $\mathcal{S} = \{\vec{u}_1, \dots, \vec{u}_m\}$ be a subset of \mathbb{R}^n . If \mathcal{S} is an orthogonal set which consists of nonzero vectors, then \mathcal{S} is linearly independent.
- > Know the definition of an orthogonal bases.
- > **Theorem.** (Easy expansions with orthogonal bases.) Let $\{\vec{u}_1, \dots, \vec{u}_m\}$ be an orthogonal basis of a subspace \mathcal{W} of \mathbb{R}^n . Then for every $\vec{y} \in \mathcal{W}$ we have

$$\vec{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 + \dots + \frac{\vec{y} \cdot \vec{u}_m}{\vec{u}_m \cdot \vec{u}_m} \vec{u}_m$$

- > Know the definition of the orthogonal projection of a vector \vec{y} onto a vector \vec{u} : A vector $\widehat{\vec{y}} = \alpha \vec{u}$ is called the **orthogonal projection of \vec{y} onto \vec{u}** if the difference $\vec{y} - \widehat{\vec{y}}$ is orthogonal to \vec{u} . (Convince yourself that

$$\widehat{\vec{y}} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$

is the orthogonal projection of \vec{y} onto \vec{u} .)

- > Know how to do calculations with orthogonal projections.
- > The definitions of orthonormal set of vectors, orthonormal basis, a matrix with orthonormal columns.
- > Know the characterization of a matrix with orthonormal columns: The columns of U are orthonormal if and only if $U^T U = I$. (Please make sure that you understand the order of the matrix and its transpose in the previous identity.)
- > Know the properties of matrices with orthonormal columns.

6.3 Orthogonal projections.

- > Know the definition of the orthogonal projection of a vector \vec{y} onto a subspace \mathcal{W} : A vector $\widehat{\vec{y}} \in \mathcal{W}$ is called the **orthogonal projection of \vec{y} onto \mathcal{W}** if the difference $\vec{y} - \widehat{\vec{y}}$ is orthogonal to \mathcal{W} .
- > **Theorem.** (The orthogonal decomposition theorem.) Let \mathcal{W} be a subspace of \mathbb{R}^n . Then each \vec{y} in \mathbb{R}^n can be written uniquely in the form

$$\vec{y} = \widehat{\vec{y}} + \vec{z}$$

where $\widehat{\vec{y}} \in \mathcal{W}$ and $\vec{z} \in \mathcal{W}^\perp$. If $\{\vec{u}_1, \dots, \vec{u}_m\}$ is an orthogonal basis for \mathcal{W} , then

$$\widehat{\vec{y}} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 + \dots + \frac{\vec{y} \cdot \vec{u}_m}{\vec{u}_m \cdot \vec{u}_m} \vec{u}_m \quad (2)$$

- > Know that equation (2) simplifies if we assume that $\{\vec{u}_1, \dots, \vec{u}_m\}$ is an **orthonormal basis** for \mathcal{W} ; then

$$\widehat{\vec{y}} = (\vec{y} \cdot \vec{u}_1) \vec{u}_1 + (\vec{y} \cdot \vec{u}_2) \vec{u}_2 + \dots + (\vec{y} \cdot \vec{u}_m) \vec{u}_m. \quad (3)$$

- > Know the amazing fact that equation (3) can be written as a matrix equation; let

$$U = [\vec{u}_1 \ \dots \ \vec{u}_m]$$

be a matrix with **orthonormal columns**, then

$$\widehat{\vec{y}} = (\vec{y} \cdot \vec{u}_1) \vec{u}_1 + (\vec{y} \cdot \vec{u}_2) \vec{u}_2 + \dots + (\vec{y} \cdot \vec{u}_m) \vec{u}_m = U U^T \vec{y}.$$

The explanation for the last equality based on the definition of the matrix multiplication is as follows:

$$(\vec{y} \cdot \vec{u}_1) \vec{u}_1 + \dots + (\vec{y} \cdot \vec{u}_m) \vec{u}_m = [\vec{u}_1 \ \dots \ \vec{u}_m] \begin{bmatrix} \vec{y} \cdot \vec{u}_1 \\ \vdots \\ \vec{y} \cdot \vec{u}_m \end{bmatrix} = U \begin{bmatrix} (\vec{u}_1)^\top \vec{y} \\ \vdots \\ (\vec{u}_m)^\top \vec{y} \end{bmatrix} = U \begin{bmatrix} (\vec{u}_1)^\top \\ \vdots \\ (\vec{u}_m)^\top \end{bmatrix} \vec{y} = U U^T \vec{y}.$$

- Know how to solve Exercise 23. Given an $m \times n$ matrix A and a vector $\vec{v} \in \mathbb{R}^n$, know how to write \vec{v} as a sum of a vector in $\text{Nul } A$ and a vector in $\text{Row } A$.

6.4 The Gram-Schmidt orthogonalization.

- Know the Gram-Schmidt orthogonalization process: Let m and n be positive integers such that $2 \leq m \leq n$. Let $\{\vec{x}_1, \dots, \vec{x}_m\}$ be a basis for a subspace \mathcal{W} of \mathbb{R}^n . The vectors $\vec{v}_1, \dots, \vec{v}_m$ recursively defined by

$$\begin{aligned}\vec{v}_1 &= \vec{x}_1, \\ \vec{v}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1, \\ \vec{v}_3 &= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2, \\ &\vdots \\ \vec{v}_m &= \vec{x}_m - \frac{\vec{x}_m \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_m \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 - \dots - \frac{\vec{x}_m \cdot \vec{v}_{m-1}}{\vec{v}_{m-1} \cdot \vec{v}_{m-1}} \vec{v}_{m-1},\end{aligned}$$

have the following properties

- (i) $\{\vec{v}_1, \dots, \vec{v}_m\}$ is an orthogonal basis for \mathcal{W} .
 - (ii) For all $k \in \{1, \dots, m\}$ we have $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{x}_1, \dots, \vec{x}_k\}$.
- Know the definition and how to construct a QR factorization of a matrix with linearly independent columns.

6.5 Least square problems.

- Know the definition of a **least-squares solution** of $A\vec{x} = \vec{b}$.
- Know the theorem stating the connection between the set of least-squares solutions of $A\vec{x} = \vec{b}$ and the set of solutions of the normal equations $A^\top A\vec{x} = A^\top \vec{b}$.
- Know the necessary and sufficient condition for the uniqueness of the least-squares solution of $A\vec{x} = \vec{b}$ (and its proof).
- Know how to find the least-squares solution of $A\vec{x} = \vec{b}$ using the QR factorization of A .
- Know the proofs of the theorem and its corollaries posted on May 2, 2016.

6.6 Applications to linear models.

- Know how to find the least-squares line for a set of data.
- Know how to find the least-squares fitting for other curves.
- Know how to find the least-squares plane for a set of data.
- Know how to solve Exercise 14.

6.7 Inner product spaces.

- Know the definition of an **inner product**.
- Know the definitions of length, distance and orthogonality in an inner product space.
- Know how to find the best approximation in an inner product space.
- Know the statement and the proof of the Cauchy-Schwarz inequality.
- Know the statement and the proof of the triangle inequality in an inner product space.

6.8 Applications of inner product spaces.

- Know how to solve weighted least-squares problems (Example 1, Exercises 1, 2).
- Know how to find quadratic and cubic data trend for a given set of data (Example 2, Exercises 3, 4).

7.1 Diagonalization of symmetric matrices.

- Know the theorem about the orthogonality of eigenvectors corresponding to distinct eigenvalues of a symmetric matrix and how to prove it.
- Know the definition of an orthogonally diagonalizable matrix.
- Know the relationship between the concepts of a symmetric and an orthogonally diagonalizable matrix and how to prove the easier direction of this relationship.
- Know how to prove that a symmetric 2×2 matrix is orthogonally diagonalizable.
- Know the spectral decomposition formula (and its meaning) for symmetric matrices: Let $A = UDU^\top$ be an orthogonal diagonalization of a symmetric matrix A . Then

$$\begin{aligned} A = UDU^\top &= [\vec{u}_1 \ \vec{u}_2 \ \cdots \ \vec{u}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{u}_1^\top \\ \vec{u}_2^\top \\ \vdots \\ \vec{u}_n^\top \end{bmatrix} \\ &= [\lambda_1 \vec{u}_1 \ \lambda_2 \vec{u}_2 \ \cdots \ \lambda_n \vec{u}_n] \begin{bmatrix} \vec{u}_1^\top \\ \vec{u}_2^\top \\ \vdots \\ \vec{u}_n^\top \end{bmatrix} = \lambda_1 \vec{u}_1 \vec{u}_1^\top + \lambda_2 \vec{u}_2 \vec{u}_2^\top + \cdots + \lambda_n \vec{u}_n \vec{u}_n^\top \end{aligned}$$

(Here, for $k \in \{1, \dots, n\}$ the $n \times n$ matrix $\vec{u}_k \vec{u}_k^\top$ is the orthogonal projection matrix onto the unit vector \vec{u}_k .)