

On an exam you can be asked to prove any of the theorems stated in the summary.

4.7 Change of bases (in fact: Change of coordinates).

- > Know that the matrix M with the property $[\mathbf{v}]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **change-of-coordinate matrix from \mathcal{B} to \mathcal{C}** . It is denoted $P_{\mathcal{C} \leftarrow \mathcal{B}}$ and it is calculated as

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix};$$

here $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and \mathcal{C} are two bases of an n -dimensional vector space \mathcal{V} and $[\mathbf{v}]_{\mathcal{B}} \in \mathbb{R}^n$ denotes the coordinate vector of $\mathbf{v} \in \mathcal{V}$ relative to the basis \mathcal{B} .

- > $\left(P_{\mathcal{C} \leftarrow \mathcal{B}}\right)^{-1} = P_{\mathcal{B} \leftarrow \mathcal{C}}$
- > Know that there is a special basis of \mathbb{R}^n , called the **standard basis**, which consists of the columns of the identity matrix I_n . It is denoted by \mathcal{E} . This notation comes from the fact that the vectors in this basis are commonly denoted by $\vec{e}_1, \dots, \vec{e}_n$.
- > Know that for a basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ for \mathbb{R}^n (this is a special case) we have

$$P_{\mathcal{E} \leftarrow \mathcal{B}} = [\vec{b}_1 \cdots \vec{b}_n] = P_{\mathcal{B}}$$

(the matrix $P_{\mathcal{B}}$ was introduced in Section 4.4); this basically says that $P_{\mathcal{E} \leftarrow \mathcal{B}}$ is a very friendly matrix.

- > Know $P_{\mathcal{C} \leftarrow \mathcal{B}} = \left(P_{\mathcal{C} \leftarrow \mathcal{E}}\right) \left(P_{\mathcal{E} \leftarrow \mathcal{B}}\right) = \left(P_{\mathcal{E} \leftarrow \mathcal{C}}\right)^{-1} P_{\mathcal{E} \leftarrow \mathcal{B}} = [\vec{c}_1 \cdots \vec{c}_n]^{-1} [\vec{b}_1 \cdots \vec{b}_n]$ where $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$ is another basis for \mathbb{R}^n ; this basically says that $P_{\mathcal{C} \leftarrow \mathcal{B}}$ can be written as a product of a very friendly matrix and the inverse of another very friendly matrix
- > Know Exercises 4 - 10, 13, 14.

5.1 Eigenvectors and eigenvalues.

- > Know the definition of an eigenvector and eigenvalue. It is a little tricky. Pay attention.
- > Know the definition of an eigenspace and how to find an eigenspace corresponding to a given eigenvalue.
- > Know that the eigenvalues of a triangular matrix are the entries on its main diagonal.
- > **Theorem.** Eigenvectors corresponding to distinct eigenvalues are linearly independent. (Or, in more formal mathematical language: Let A be an $n \times n$ matrix, let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \in \mathbb{R}^n$ and $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$. If $A\vec{v}_k = \lambda_k \vec{v}_k$, $\vec{v}_k \neq \vec{0}$ and $\lambda_j \neq \lambda_k$ for all $j, k = 1, 2, \dots, m$, then $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ are linearly independent.
- > Know the proof of the above theorem for $m = 2$ vectors.

5.2 The characteristic equation.

- > Know that λ is an eigenvalue of an $n \times n$ matrix A if and only if $\det(A - \lambda I) = 0$
- > Know how to calculate $\det(A - \lambda I)$ for 2×2 matrices, how to find eigenvalues and corresponding eigenvectors. Exercises 1-8, but do more and find eigenvectors as well.

5.3 Diagonalization.

- > **Theorem.** (The diagonalization theorem) An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
- > Know how to decide whether a given 2×2 and 3×3 matrix A , is diagonalizable or not; if it is diagonalizable, how to find invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

5.4 Eigenvectors and linear transformations.

- Know that the matrix M with the property $[T\mathbf{v}]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **matrix of T relative to the bases \mathcal{B} and \mathcal{C}** and it is calculated as

$$M = \begin{bmatrix} [T\mathbf{b}_1]_{\mathcal{C}} & \cdots & [T\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix},$$

where T be a linear transformation from an m -dimensional vector space \mathcal{V} to an n -dimensional vector space \mathcal{W} , $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ is a basis for \mathcal{V} and \mathcal{C} is a basis for \mathcal{W} .

- Know that the matrix M with the property $[T\mathbf{v}]_{\mathcal{B}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **matrix of T relative to the basis \mathcal{B}** , it is denoted by $[T]_{\mathcal{B}}$ and it is calculated as

$$[T]_{\mathcal{B}} = \begin{bmatrix} [T\mathbf{b}_1]_{\mathcal{B}} & \cdots & [T\mathbf{b}_n]_{\mathcal{B}} \end{bmatrix},$$

where $T : \mathcal{V} \rightarrow \mathcal{V}$ is a linear transformation on an n -dimensional vector space \mathcal{V} and $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for \mathcal{V} .

- **Theorem.** (The diagonal matrix representation) Let A, D, P be $n \times n$ matrices, where P is invertible and D is diagonal. If \mathcal{B} is the basis for \mathbb{R}^n which consists of the columns of P and if $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given by $T\vec{v} = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^n$, then $[T]_{\mathcal{B}} = D$.

5.5 Complex eigenvalues.

- Know how to calculate complex eigenvalues and corresponding eigenvectors of 2×2 and 3×3 real matrices
- Know that the most important class of 2×2 matrices with complex eigenvalues are rotation matrices; the matrix of the counterclockwise rotation about the origin by the angle θ measured in radians relative to the standard basis for \mathbb{R}^2 is

$$R_{\theta} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix};$$

the 2×2 matrix R_{θ} is called the **rotation matrix**.

- **Theorem.** (A “hiding rotation” theorem.) Let A be a real 2×2 matrix with a nonreal eigenvalue. Then there exist an invertible 2×2 matrix P , a positive scalar α and a rotation matrix R_{θ} such that $A = \alpha P R_{\theta} P^{-1}$.

6.1 Inner product, length, and orthogonality.

- Know the definition of the inner product (dot product) in \mathbb{R}^n , its basic properties and calculations involving it.
- Know the definition, the basic properties of the length in \mathbb{R}^n and calculations involving it.
- Know the definition of the distance in \mathbb{R}^n and calculations involving it.
- Know the definition of orthogonality in \mathbb{R}^n and calculations involving it.
- Know the statement and the proof of the linear algebra version of **Pythagorean theorem**.
- Know the definition and the basic properties of the orthogonal complement in \mathbb{R}^n .
- Know that for a given $m \times n$ matrix A we have $(\text{Row } A)^{\perp} = \text{Nul } A$ and $(\text{Col } A)^{\perp} = \text{Nul } A^{\top}$.
- Know the geometric interpretation of the inner product (dot product) in \mathbb{R}^2 and \mathbb{R}^3 :

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \vartheta, \tag{1}$$

where \vec{u} and \vec{v} are vectors in \mathbb{R}^2 or in \mathbb{R}^3 , ϑ is the angle at the vertex O in the triangle OAB with O being the origin, A being the endpoint of \vec{u} and B the endpoint of \vec{v} . (You should know the proof of formula (1).)

6.2 Orthogonal sets.

- Know the definition of an orthogonal set of vectors.
- **Theorem.** (Linear independence of orthogonal sets.) Let $\mathcal{S} = \{\vec{u}_1, \dots, \vec{u}_m\}$ be a subset of \mathbb{R}^n . If \mathcal{S} is an orthogonal set which consists of nonzero vectors, then \mathcal{S} is linearly independent.
- Know the definition of an orthogonal bases.
- **Theorem.** (Easy expansions with orthogonal bases.) Let $\{\vec{u}_1, \dots, \vec{u}_m\}$ be an orthogonal basis of a subspace \mathcal{W} of \mathbb{R}^n . Then for every $\vec{y} \in \mathcal{W}$ we have

$$\vec{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 + \dots + \frac{\vec{y} \cdot \vec{u}_m}{\vec{u}_m \cdot \vec{u}_m} \vec{u}_m$$

- Know the definition of the orthogonal projection of a vector \vec{y} onto a vector \vec{u} : A vector $\widehat{\vec{y}} = \alpha \vec{u}$ is called the **orthogonal projection of \vec{y} onto \vec{u}** if the difference $\vec{y} - \widehat{\vec{y}}$ is orthogonal to \vec{u} . (Convince yourself that

$$\widehat{\vec{y}} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$

is the orthogonal projection of \vec{y} onto \vec{u} .)

- Know how to do calculations with orthogonal projections.
- The definitions of orthonormal set of vectors, orthonormal basis, a matrix with orthonormal columns.
- Know the characterization of a matrix with orthonormal columns: The columns of U are orthonormal if and only if $U^T U = I$. (Please make sure that you understand the order of the matrix and its transpose in the previous identity.)
- Know the properties of matrices with orthonormal columns.

6.3 Orthogonal projections.

- Know the definition of the orthogonal projection of a vector \vec{y} onto a subspace \mathcal{W} : A vector $\widehat{\vec{y}} \in \mathcal{W}$ is called the **orthogonal projection of \vec{y} onto \mathcal{W}** if the difference $\vec{y} - \widehat{\vec{y}}$ is orthogonal to \mathcal{W} .
- **Theorem.** (The orthogonal decomposition theorem.) Let \mathcal{W} be a subspace of \mathbb{R}^n . Then each \vec{y} in \mathbb{R}^n can be written uniquely in the form

$$\vec{y} = \widehat{\vec{y}} + \vec{z}$$

where $\widehat{\vec{y}} \in \mathcal{W}$ and $\vec{z} \in \mathcal{W}^\perp$. If $\{\vec{u}_1, \dots, \vec{u}_m\}$ is an orthogonal basis for \mathcal{W} , then

$$\widehat{\vec{y}} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 + \dots + \frac{\vec{y} \cdot \vec{u}_m}{\vec{u}_m \cdot \vec{u}_m} \vec{u}_m \quad (2)$$

- Know that equation (2) simplifies if we assume that $\{\vec{u}_1, \dots, \vec{u}_m\}$ is an **orthonormal basis** for \mathcal{W} ; then

$$\widehat{\vec{y}} = (\vec{y} \cdot \vec{u}_1) \vec{u}_1 + (\vec{y} \cdot \vec{u}_2) \vec{u}_2 + \dots + (\vec{y} \cdot \vec{u}_m) \vec{u}_m. \quad (3)$$

- Know the amazing fact that equation (3) can be written as a matrix equation; let

$$U = [\vec{u}_1 \ \dots \ \vec{u}_m]$$

be a matrix with **orthonormal columns**, then

$$\widehat{\vec{y}} = (\vec{y} \cdot \vec{u}_1) \vec{u}_1 + (\vec{y} \cdot \vec{u}_2) \vec{u}_2 + \dots + (\vec{y} \cdot \vec{u}_m) \vec{u}_m = U U^T \vec{y}.$$

The explanation for the last equality based on the definition of the matrix multiplication is as follows:

$$(\vec{y} \cdot \vec{u}_1) \vec{u}_1 + \dots + (\vec{y} \cdot \vec{u}_m) \vec{u}_m = [\vec{u}_1 \ \dots \ \vec{u}_m] \begin{bmatrix} \vec{y} \cdot \vec{u}_1 \\ \vdots \\ \vec{y} \cdot \vec{u}_m \end{bmatrix} = U \begin{bmatrix} (\vec{u}_1)^T \vec{y} \\ \vdots \\ (\vec{u}_m)^T \vec{y} \end{bmatrix} = U \begin{bmatrix} (\vec{u}_1)^T \\ \vdots \\ (\vec{u}_m)^T \end{bmatrix} \vec{y} = U U^T \vec{y}.$$

- Know how to solve Exercise 23. Given an $m \times n$ matrix A and a vector $\vec{v} \in \mathbb{R}^n$, know how to write \vec{v} as a sum of a vector in $\text{Nul } A$ and a vector in $\text{Row } A$.

6.4 The Gram-Schmidt orthogonalization.

- Know the Gram-Schmidt orthogonalization process: Let m and n be positive integers such that $2 \leq m \leq n$. Let $\{\vec{x}_1, \dots, \vec{x}_m\}$ be a basis for a subspace \mathcal{W} of \mathbb{R}^n . The vectors $\vec{v}_1, \dots, \vec{v}_m$ recursively defined by

$$\begin{aligned}\vec{v}_1 &= \vec{x}_1, \\ \vec{v}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1, \\ \vec{v}_3 &= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2, \\ &\vdots \\ \vec{v}_m &= \vec{x}_m - \frac{\vec{x}_m \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_m \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 - \dots - \frac{\vec{x}_m \cdot \vec{v}_{m-1}}{\vec{v}_{m-1} \cdot \vec{v}_{m-1}} \vec{v}_{m-1},\end{aligned}$$

have the following properties

- (i) $\{\vec{v}_1, \dots, \vec{v}_m\}$ is an orthogonal basis for \mathcal{W} .
 - (ii) For all $k \in \{1, \dots, m\}$ we have $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\{\vec{x}_1, \dots, \vec{x}_k\}$.
- Know the definition and how to construct a QR factorization of a matrix with linearly independent columns.

6.5 Least square problems.

- Know the definition of a **least-squares solution** of $A\vec{x} = \vec{b}$.
- Know the theorem stating the connection between the set of least-squares solutions of $A\vec{x} = \vec{b}$ and the set of solutions of the normal equations $A^T A\vec{x} = A^T \vec{b}$.
- Know the necessary and sufficient condition for the uniqueness of the least-squares solution of $A\vec{x} = \vec{b}$ (and its proof).
- Know how to find the least-squares solution of $A\vec{x} = \vec{b}$ using the QR factorization of A .
- Know the proofs of the theorem and its corollaries posted on May 2, 2016.

6.6 Applications to linear models.

- Know how to find the least-squares line for a set of data.
- Know how to find the least-squares fitting for other curves.
- Know how to find the least-squares plane for a set of data.
- Know how to solve Exercise 14.

6.7 Inner product spaces.

- Know the definition of an **inner product**.
- Know the definitions of length, distance and orthogonality in an inner product space.
- Know the statement and the proof of the abstract Pythagorean theorem.
- Know how to find the best approximation in an inner product space.
- Know the statement and the proof of the Cauchy-Schwarz inequality.
- Know the statement and the proof of the triangle inequality in an inner product space.

6.8 Applications of inner product spaces.

- Know how to solve weighted least-squares problems (Example 1, Exercises 1, 2).
- Know how to find quadratic and cubic data trend for a given set of data (Example 2, Exercises 3, 4).

7.1 Diagonalization of symmetric matrices.

- Know the theorem about the orthogonality of eigenvectors corresponding to distinct eigenvalues of a symmetric matrix and how to prove it.
- Know the definition of an orthogonally diagonalizable matrix.
- Know the relationship between the concepts of a symmetric and an orthogonally diagonalizable matrix and how to prove the easier direction of this relationship.
- Know how to prove that a symmetric 2×2 matrix is orthogonally diagonalizable.
- Know the spectral decomposition formula (and its meaning) for symmetric matrices: Let $A = UDU^\top$ be an orthogonal diagonalization of a symmetric matrix A . Then

$$\begin{aligned}
 A = UDU^\top &= [\vec{u}_1 \ \vec{u}_2 \ \cdots \ \vec{u}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{u}_1^\top \\ \vec{u}_2^\top \\ \vdots \\ \vec{u}_n^\top \end{bmatrix} \\
 &= [\lambda_1 \vec{u}_1 \ \lambda_2 \vec{u}_2 \ \cdots \ \lambda_n \vec{u}_n] \begin{bmatrix} \vec{u}_1^\top \\ \vec{u}_2^\top \\ \vdots \\ \vec{u}_n^\top \end{bmatrix} = \lambda_1 \vec{u}_1 \vec{u}_1^\top + \lambda_2 \vec{u}_2 \vec{u}_2^\top + \cdots + \lambda_n \vec{u}_n \vec{u}_n^\top
 \end{aligned}$$

(Here, for $k \in \{1, \dots, n\}$ the $n \times n$ matrix $\vec{u}_k \vec{u}_k^\top$ is the orthogonal projection matrix onto the unit vector \vec{u}_k .)

7.2 Quadratic forms.

- Know the definition of a quadratic form.
- Know how to transform a quadratic form into a quadratic form with only square terms.
- Know how to classify quadratic forms, including positive semidefinite and negative semidefinite forms.

7.3 Constrained optimization.

- Know how to solve problems like Example 3 (also including the minimum value).

7.4 The singular value decomposition.

- Know the definition of the singular value decomposition of a real $m \times n$ matrix A : A **singular value decomposition** of a real $m \times n$ matrix A is a factorization of the form

$$A = U\Sigma V^\top$$

where U is $m \times m$ orthogonal matrix, V is $n \times n$ and Σ is $m \times n$ matrix of the form

$$\Sigma = \left[\begin{array}{ccc|c} \sigma_1 & \cdots & 0 & \mathbf{0}_{r \times (n-r)} \\ \vdots & \ddots & \vdots & \\ 0 & \cdots & \sigma_r & \\ \hline \mathbf{0}_{(m-r) \times r} & & & \mathbf{0}_{(m-r) \times (n-r)} \end{array} \right]$$

where $r = \text{rank } A$, $\begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix}$ is $r \times r$ diagonal matrix with positive entries on the diagonal and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and the remaining entries of Σ are zeros. The values $\sigma_1, \sigma_2, \dots, \sigma_r$ are called the **singular values** of A . The columns of V are called **right singular vectors** of A and the columns of U are called **left singular vectors** of A .

- Know the consequences of the definition of a singular value decomposition. (For example, $A^\top = V\Sigma^\top U^\top$, $A^\top A = V\Sigma^\top \Sigma V^\top$, where $\Sigma^\top \Sigma$ is $n \times n$ diagonal matrix with the eigenvalues of $A^\top A$ on the diagonal and the positive entries on the diagonal are equal to $\sigma_1^2, \dots, \sigma_r^2$.)
- Know how a singular value decomposition of A contains information about orthonormal bases for all four fundamental subspaces associated with A . This is summarized in Figure 4 on page 479.