

### 6.1 Inner product, length, and orthogonality.

- Know the definition of the dot product in  $\mathbb{R}^n$ , its basic properties and calculations involving it.
- Know the definition, the basic properties of the length of a vector in  $\mathbb{R}^n$ , its properties and calculations involving it.
- Know the definition of the distance in  $\mathbb{R}^n$  and calculations involving it.
- Know the definition of orthogonality in  $\mathbb{R}^n$  and calculations involving it.
- Know the statement and the proof of the linear algebra version of **Pythagorean theorem**.
- Know the definition and the basic properties of the orthogonal complement in  $\mathbb{R}^n$ .
- Know that for a given  $m \times n$  matrix  $A$  we have  $(\text{Row } A)^\perp = \text{Nul } A$  and  $(\text{Col } A)^\perp = \text{Nul } A^\top$ .
- Know the geometric interpretation of the dot product in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ :

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \vartheta, \tag{1}$$

where  $\mathbf{u}$  and  $\mathbf{v}$  are vectors in  $\mathbb{R}^2$  or in  $\mathbb{R}^3$ ,  $\vartheta$  is the angle at the vertex  $O$  in the triangle  $OAB$  with  $O$  being the origin,  $A$  being the endpoint of  $\mathbf{u}$  and  $B$  the endpoint of  $\mathbf{v}$ . (You should know the proof of formula (1).)

### 6.2 Orthogonal sets.

- Know the definition of an orthogonal set of vectors.
- **Theorem.** (Linear independence of orthogonal sets.) Let  $\mathcal{S} = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$  be a subset of  $\mathbb{R}^n$ . If  $\mathcal{S}$  is an orthogonal set which consists of nonzero vectors, then  $\mathcal{S}$  is linearly independent.
- Know the definition of an orthogonal bases.
- **Theorem.** (Easy expansions with orthogonal bases.) Let  $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$  be an orthogonal basis of a subspace  $\mathcal{W}$  of  $\mathbb{R}^n$ . Then for every  $\mathbf{y} \in \mathcal{W}$  we have

$$\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 + \dots + \frac{\mathbf{y} \cdot \mathbf{u}_m}{\mathbf{u}_m \cdot \mathbf{u}_m} \mathbf{u}_m$$

- Know the definition of the orthogonal projection of a vector  $\mathbf{y}$  onto a vector  $\mathbf{u}$ : A vector  $\hat{\mathbf{y}} = \alpha \mathbf{u}$  is called the **orthogonal projection of  $\mathbf{y}$  onto  $\mathbf{u}$**  if the difference  $\mathbf{y} - \hat{\mathbf{y}}$  is orthogonal to  $\mathbf{u}$ . (Convince yourself that

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$$

is the orthogonal projection of  $\mathbf{y}$  onto  $\mathbf{u}$ .)

- Know how to do calculations with orthogonal projections.
- The definitions of orthonormal set of vectors, orthonormal basis, a matrix with orthonormal columns.
- Know the characterization of a matrix with orthonormal columns: The columns of  $n \times m$  matrix  $U$  are orthonormal if and only if  $U^\top U = I_m$ . (Please make sure that you understand the order of the matrix and its transpose in the previous identity.)
- Know the properties of matrices with orthonormal columns.

### 6.3 Orthogonal projections.

- Know the definition of the orthogonal projection of a vector  $\mathbf{y}$  onto a subspace  $\mathcal{W}$ : A vector  $\hat{\mathbf{y}} \in \mathcal{W}$  is called the **orthogonal projection of  $\mathbf{y}$  onto  $\mathcal{W}$**  if the difference  $\mathbf{y} - \hat{\mathbf{y}}$  is orthogonal to  $\mathcal{W}$ . The orthogonal projection of the vector  $\mathbf{y}$  onto a subspace  $\mathcal{W}$  is denoted by  $\text{Proj}_{\mathcal{W}} \mathbf{y}$ .

- **Theorem.** (The orthogonal decomposition theorem.) Let  $\mathcal{W}$  be a subspace of  $\mathbb{R}^n$ . Then each  $\mathbf{y}$  in  $\mathbb{R}^n$  can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where  $\hat{\mathbf{y}} \in \mathcal{W}$  and  $\mathbf{z} \in \mathcal{W}^\perp$ . We have that  $\hat{\mathbf{y}} = \text{Proj}_{\mathcal{W}} \mathbf{y}$ . If  $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$  is an orthogonal basis for  $\mathcal{W}$ , then

$$\text{Proj}_{\mathcal{W}} \mathbf{y} = \hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 + \dots + \frac{\mathbf{y} \cdot \mathbf{u}_m}{\mathbf{u}_m \cdot \mathbf{u}_m} \mathbf{u}_m \quad (2)$$

- Know that equation (2) simplifies if we assume that  $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$  is an **orthonormal basis** for  $\mathcal{W}$ ; then

$$\text{Proj}_{\mathcal{W}} \mathbf{y} = \hat{\mathbf{y}} = (\mathbf{y} \cdot \mathbf{u}_1) \mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2) \mathbf{u}_2 + \dots + (\mathbf{y} \cdot \mathbf{u}_m) \mathbf{u}_m. \quad (3)$$

- Know the amazing fact that equation (3) can be written as a matrix equation; let

$$U = [\mathbf{u}_1 \ \dots \ \mathbf{u}_m]$$

be a matrix with **orthonormal columns**, then

$$\text{Proj}_{\text{Col } Q} \mathbf{y} = \hat{\mathbf{y}} = (\mathbf{y} \cdot \mathbf{u}_1) \mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2) \mathbf{u}_2 + \dots + (\mathbf{y} \cdot \mathbf{u}_m) \mathbf{u}_m = U U^\top \mathbf{y}.$$

The explanation for the last equality based on the definition of the matrix multiplication is as follows:

$$(\mathbf{y} \cdot \mathbf{u}_1) \mathbf{u}_1 + \dots + (\mathbf{y} \cdot \mathbf{u}_m) \mathbf{u}_m = [\mathbf{u}_1 \ \dots \ \mathbf{u}_m] \begin{bmatrix} \mathbf{y} \cdot \mathbf{u}_1 \\ \vdots \\ \mathbf{y} \cdot \mathbf{u}_m \end{bmatrix} = U \begin{bmatrix} (\mathbf{u}_1)^\top \mathbf{y} \\ \vdots \\ (\mathbf{u}_m)^\top \mathbf{y} \end{bmatrix} = U \begin{bmatrix} (\mathbf{u}_1)^\top \\ \vdots \\ (\mathbf{u}_m)^\top \end{bmatrix} \mathbf{y} = U U^\top \mathbf{y}.$$

- Know how to prove the following fact: Let  $Q$  be a  $n \times m$  matrix with orthonormal columns. Let  $\mathbf{y} \in \mathbb{R}^n$ . Prove that the projection of  $\mathbf{y}$  onto the column space of  $Q$  is given by the formula  $QQ^\top \mathbf{y}$ .

#### 6.4 The Gram-Schmidt orthogonalization.

- Know the Gram-Schmidt orthogonalization process: Let  $m$  and  $n$  be positive integers such that  $2 \leq m \leq n$ . Let  $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$  be a basis for a subspace  $\mathcal{W}$  of  $\mathbb{R}^n$ . The vectors  $\mathbf{v}_1, \dots, \mathbf{v}_m$  recursively defined by

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{x}_1, \\ \mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1, \\ \mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2, \\ &\vdots \\ \mathbf{v}_m &= \mathbf{x}_m - \frac{\mathbf{x}_m \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_m \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 - \dots - \frac{\mathbf{x}_m \cdot \mathbf{v}_{m-1}}{\mathbf{v}_{m-1} \cdot \mathbf{v}_{m-1}} \mathbf{v}_{m-1}, \end{aligned}$$

have the following properties

- $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$  is an orthogonal basis for  $\mathcal{W}$ .
- For all  $k \in \{1, \dots, m\}$  we have  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ .

- Know the definition and how to construct a  $QR$  factorization of a matrix with linearly independent columns.

#### 6.5 Least square problems.

- Know the definition of a **least-squares solution** of  $A\mathbf{x} = \mathbf{b}$ .
- Know the theorem stating the connection between the set of least-squares solutions of  $A\mathbf{x} = \mathbf{b}$  and the set of solutions of the normal equations  $A^\top A\mathbf{x} = A^\top \mathbf{b}$ .

- Know the necessary and sufficient condition for the uniqueness of the least-squares solution of  $A\mathbf{x} = \mathbf{b}$  (and its proof).
- Know how to find the least-squares solution of  $A\mathbf{x} = \mathbf{b}$  using the  $QR$  factorization of  $A$ .
- Know how to prove the following statement: The matrices  $A$  and  $A^T A$  have the same null space.

## 6.6 Applications to linear models.

- Know how to find the least-squares line for a set of data points.
- Know how to find the least-squares fitting for other curves.
- Know how to find the least-squares plane for a set of data points.

## 7.1 Diagonalization of symmetric matrices.

- Know the theorem about the orthogonality of eigenvectors corresponding to distinct eigenvalues of a symmetric matrix and how to prove it.
- Know the definition of an orthogonally diagonalizable matrix.
- Know the relationship between the concepts of a symmetric and an orthogonally diagonalizable matrix and how to prove the easier direction of this relationship.
- Know how to prove that the eigenvalues of a symmetric  $2 \times 2$  matrix are real.
- Know the spectral decomposition formula (and its meaning) for symmetric matrices: Let  $A = UDU^T$  be an orthogonal diagonalization of a symmetric matrix  $A$ . Then

$$\begin{aligned}
 A = UDU^T &= [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \vdots \\ \mathbf{u}_n^\top \end{bmatrix} \\
 &= [\lambda_1 \mathbf{u}_1 \quad \lambda_2 \mathbf{u}_2 \quad \cdots \quad \lambda_n \mathbf{u}_n] \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \vdots \\ \mathbf{u}_n^\top \end{bmatrix} = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^\top + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^\top + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^\top
 \end{aligned}$$

(Here, for  $k \in \{1, \dots, n\}$  the  $n \times n$  matrix  $\mathbf{u}_k \mathbf{u}_k^\top$  is the orthogonal projection matrix onto the unit vector  $\mathbf{u}_k$ .)

## 7.2 Quadratic forms.

- Know the definition of a quadratic form.
- Know how to transform a quadratic form into a quadratic form with only square terms.
- Know how to classify quadratic forms, including positive semidefinite and negative semidefinite forms.