Winter 2019 Math 304 Topics for Final Exam

4.7 Change of bases (in fact: Change of coordinates).

➢ Know that the matrix M with the property $[\mathbf{v}]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$**. It is denoted $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and it is calculated as

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \Big[\, [\mathbf{b}_1]_{\mathcal{C}} \ \cdots \ [\mathbf{b}_n]_{\mathcal{C}} \,\Big];$$

here $\mathcal{B} = \{\mathbf{b}, \ldots, \mathbf{b}_n\}$ and $\mathcal{C}$ are two bases of an n-dimensional vector space $\mathcal{V}$ and $[\mathbf{v}]_{\mathcal{B}} \in \mathbb{R}^n$ denotes the coordinate vector of $\mathbf{v} \in \mathcal{V}$ relative to the basis $\mathcal{B}$.

➢ $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$

➢ Know that there is a special basis of $\mathbb{R}^n$, called the **standard basis**, which consists of the columns of the identify matrix I_n. It is denoted by $\mathcal{E}$. This notattion comes from the fact that the vectors in this basis are commonly denoted by $\mathbf{e}_1, \cdots, \mathbf{e}_n$.

➢ Know that for a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ for $\mathbb{R}^n$ (this is a special case) we have

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \big[\, \mathbf{b}_1 \cdots \mathbf{b}_n \,\big] = P_{\mathcal{B}}$$

(the matrix $P_{\mathcal{B}}$ was introduced in Section 4.4); this basically says that $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ is a very friendly matrix.

➢ Know $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{E}}{P}\right)\left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\right) = \left(\underset{\mathcal{E}\leftarrow\mathcal{C}}{P}\right)^{-1} \underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \big[\, \mathbf{c}_1 \cdots \mathbf{c}_n \,\big]^{-1} \big[\, \mathbf{b}_1 \cdots \mathbf{b}_n \,\big]$
where $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ is another basis for $\mathbb{R}^n$; this basically says that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ can be written as a product of a very friendly matrix and the inverse of another very friendly matrix

➢ Know Exercises 4 - 10, 13, 14.

5.1 Eigenvectors and eigenvalues.

➢ Know the definition of an eigenvector and eigenvalue. It is a little tricky. Pay attention.

➢ Know the definition of an eigenspace and how to find an eigenspace corresponding to a given eigenvalue.

➢ Know that the eigenvalues of a triangular matrix are the entries on its main diagonal.

➢ **Theorem.** Eigenvectors corresponding to distinct eigenvalues are linearly independent. (Or, in a more formal mathematical language: Let A be an $n \times n$ matrix, let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ and $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$. If $A\mathbf{v}_k = \lambda_k \mathbf{v}_k$, $\mathbf{v}_k \neq \mathbf{0}$ and $\lambda_j \neq \lambda_k$ for all $j, k = 1, 2, \ldots, m$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent.

➢ Know the proof of the above theorem for $m = 2$ vectors.

5.2 The characteristic equation.

➢ Know that λ is an eigenvalue of an $n \times n$ matrix A if and only if $\det(A - \lambda I) = 0$

➢ Know how to calculate $\det(A - \lambda I)$ for 2×2 matrices, how to find eigenvalues and corresponding eigenvectors. Exercises 1-8, but do more and find eigenvectors as well.

5.3 Diagonalization.

➢ **Theorem.** (The diagonalization theorem) An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

➢ Know how to decide whether a given 2×2 and 3×3 matrix A, is diagonalizable or not; if it is diagonalizable, how to find invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

5.4 Eigenvectors and linear transformations.

- Know that the matrix M with the property $[T\mathbf{v}]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **matrix of T relative to the bases $\mathcal{B}$ and $\mathcal{C}$** and it is calculated as

$$M = \Big[\ [T\mathbf{b}_1]_{\mathcal{C}}\ \cdots\ [T\mathbf{b}_n]_{\mathcal{C}}\ \Big],$$

  where T be a linear transformation from an m-dimensional vector space $\mathcal{V}$ to an n-dimensional vector space $\mathcal{W}$, $\mathcal{B} = \{\mathbf{b}, \ldots, \mathbf{b}_m\}$ is a basis for $\mathcal{V}$ and $\mathcal{C}$ is a basis for $\mathcal{W}$.
- Know that the matrix M with the property $[T\mathbf{v}]_{\mathcal{B}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **matrix of T relative to the basis $\mathcal{B}$**, it is denoted by $[T]_{\mathcal{B}}$ and it is calculated as

$$[T]_{\mathcal{B}} = \Big[\ [T\mathbf{b}_1]_{\mathcal{B}}\ \cdots\ [T\mathbf{b}_n]_{\mathcal{B}}\ \Big],$$

  where $T : \mathcal{V} \to \mathcal{V}$ is a linear transformation on an n-dimensional vector space $\mathcal{V}$ and $\mathcal{B} = \{\mathbf{b}, \ldots, \mathbf{b}_n\}$ is a basis for $\mathcal{V}$.
- **Theorem.** (The diagonal matrix representation) Let A, D, P be $n\times n$ matrices, where P is invertible, D is diagonal and $A = PDP^{-1}$. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ which consists of the columns of P and if $T : \mathbb{R}^n \to \mathbb{R}^n$ is given by $T\mathbf{v} = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$, then $[T]_{\mathcal{B}} = D$.

5.5 Complex eigenvalues.

- Know how to calculate complex eigenvalues and corresponding eigenvectors of 2×2 and 3×3 real matrices
- Know that the most important class of 2×2 matrices with complex eigenvalues are rotation matrices; the matrix of the counterclockwise rotation about the origin by the angle θ measured in radians relative to the standard basis for $\mathbb{R}^2$ is

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix};$$

  the 2×2 matrix R_θ is called the **rotation matrix**.
- **Theorem.** (A "hiding rotation" theorem.) Let A be a real 2×2 matrix with a nonreal eigenvalue. Then there exist an invertible 2×2 matrix P, a positive scalar α and a rotation matrix R_θ such that $A = \alpha P R_\theta P^{-1}$.

6.1 Inner product, length, and orthogonality.

- Know the definition of the dot product in $\mathbb{R}^n$, its basic properties and calculations involving it.
- Know the definition, the basic properties of the length of a vector in $\mathbb{R}^n$, its properties and calculations involving it.
- Know the definition of the distance in $\mathbb{R}^n$ and calculations involving it.
- Know the definition of orthogonality in $\mathbb{R}^n$ and calculations involving it.
- Know the statement and the proof of the linear algebra version of **Pythagorean theorem**.
- Know the definition and the basic properties of the orthogonal complement in $\mathbb{R}^n$.
- Know that for a given $m \times n$ matrix A we have $(\operatorname{Row} A)^\perp = \operatorname{Nul} A$ and $(\operatorname{Col} A)^\perp = \operatorname{Nul}(A^\top)$.
- Know the geometric interpretation of the dot product in $\mathbb{R}^2$ and $\mathbb{R}^3$:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\| \cos\vartheta, \tag{1}$$

  where $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^2$ or in $\mathbb{R}^3$, ϑ is the angle at the vertex O in the triangle OAB with O being the origin, A being the endpoint of $\mathbf{u}$ and B the endpoint of $\mathbf{v}$. (You should know the proof of formula (1).)

6.2 Orthogonal sets.

- Know the definition of an orthogonal set of vectors.
- **Theorem.** (Linear independence of orthogonal sets.) Let $\mathcal{S} = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be a subset of $\mathbb{R}^n$. If $\mathcal{S}$ is an orthogonal set which consists of nonzero vectors, then $\mathcal{S}$ is linearly independent.
- Know the definition of an orthogonal bases.
- **Theorem.** (Easy expansions with orthogonal bases.) Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be an orthogonal basis of a subspace $\mathcal{W}$ of $\mathbb{R}^n$. Then for every $\mathbf{y} \in \mathcal{W}$ we have

$$\mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_m}{\mathbf{u}_m\cdot\mathbf{u}_m}\mathbf{u}_m$$

- Know the definition of the orthogonal projection of a vector $\mathbf{y}$ onto a vector $\mathbf{u}$: A vector $\widehat{\mathbf{y}} = \alpha\mathbf{u}$ is called the **orthogonal projection of y onto u** if the difference $\mathbf{y} - \widehat{\mathbf{y}}$ is orthogonal to $\mathbf{u}$. (Convince yourself that

$$\widehat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$$

is the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$.)
- Know how to do calculations with orthogonal projections.
- The definitions of orthonormal set of vectors, orthonormal basis, a matrix with orthonormal columns.
- Know the characterization of a matrix with orthonormal columns: The columns of $n\times m$ matrix U are orthonormal if and only if $U^\top U = I_m$. (Please make sure that you understand the order of the matrix and its transpose in the previous identity.)
- Know the properties of matrices with orthonormal columns.

6.3 Orthogonal projections.

- Know the definition of the orthogonal projection of a vector $\mathbf{y}$ onto a subspace $\mathcal{W}$: A vector $\widehat{\mathbf{y}} \in \mathcal{W}$ is called the **orthogonal projection of y onto** $\mathcal{W}$ if the difference $\mathbf{y} - \widehat{\mathbf{y}}$ is orthogonal to $\mathcal{W}$. The orthogonal projection of the vector $\mathbf{y}$ onto a subspace $\mathcal{W}$ is denoted by $\operatorname{Proj}_{\mathcal{W}}\mathbf{y}$.
- **Theorem.** (The orthogonal decomposition theorem.) Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \widehat{\mathbf{y}} + \mathbf{z}$$

where $\widehat{\mathbf{y}} \in \mathcal{W}$ and $\mathbf{z} \in \mathcal{W}^\perp$. We have that $\widehat{\mathbf{y}} = \operatorname{Proj}_{\mathcal{W}}\mathbf{y}$. If $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is an orthogonal basis for $\mathcal{W}$, then

$$\operatorname{Proj}_{\mathcal{W}}\mathbf{y} = \widehat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_m}{\mathbf{u}_m\cdot\mathbf{u}_m}\mathbf{u}_m \tag{2}$$

- Know that equation (2) simplifies if we assume that $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is an **orthonormal basis** for $\mathcal{W}$; then

$$\operatorname{Proj}_{\mathcal{W}}\mathbf{y} = \widehat{\mathbf{y}} = (\mathbf{y}\cdot\mathbf{u}_1)\,\mathbf{u}_1 + (\mathbf{y}\cdot\mathbf{u}_2)\,\mathbf{u}_2 + \cdots + (\mathbf{y}\cdot\mathbf{u}_m)\,\mathbf{u}_m. \tag{3}$$

- Know the amazing fact that equation (3) can be written as a matrix equation; let

$$U = \begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_m\end{bmatrix}$$

be a matrix with **orthonormal columns**, then

$$\operatorname{Proj}_{\operatorname{Col} Q}\mathbf{y} = \widehat{\mathbf{y}} = (\mathbf{y}\cdot\mathbf{u}_1)\,\mathbf{u}_1 + (\mathbf{y}\cdot\mathbf{u}_2)\,\mathbf{u}_2 + \cdots + (\mathbf{y}\cdot\mathbf{u}_m)\,\mathbf{u}_m = U\,U^\top\mathbf{y}.$$

The explanation for the last equality based on the definition of the matrix multiplication is as follows:

$$(\mathbf{y}\cdot\mathbf{u}_1)\,\mathbf{u}_1 + \cdots + (\mathbf{y}\cdot\mathbf{u}_m)\,\mathbf{u}_m = \begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_m\end{bmatrix}\begin{bmatrix}\mathbf{y}\cdot\mathbf{u}_1\\ \vdots\\ \mathbf{y}\cdot\mathbf{u}_m\end{bmatrix} = U\begin{bmatrix}(\mathbf{u}_1)^\top\mathbf{y}\\ \vdots\\ (\mathbf{u}_m)^\top\mathbf{y}\end{bmatrix} = U\begin{bmatrix}(\mathbf{u}_1)^\top\\ \vdots\\ (\mathbf{u}_m)^\top\end{bmatrix}\mathbf{y} = U\,U^\top\mathbf{y}.$$

- ➢ Know how to prove the following fact: Let Q be a $n \times m$ matrix with orthonormal columns. Let $\mathbf{y} \in \mathbb{R}^n$. Prove that the projection of $\mathbf{y}$ onto the column space of Q is given by the formula $QQ^T\mathbf{y}$.
- ➢ Know how to solve Exercise 23. Given an $m \times n$ matrix A and a vector $\mathbf{v} \in \mathbb{R}^n$, know how to write $\mathbf{v}$ as a sum of a vector in $\operatorname{Nul} A$ and a vector in $\operatorname{Row} A$.

6.4 The Gram-Schmidt orthogonalization.

- ➢ Know the Gram-Schmidt orthogonalization process: Let m and n be positive integers such that $2 \leq m \leq n$. Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ be a basis for a subspace $\mathcal{W}$ of $\mathbb{R}^n$. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ recursively defined by

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1, \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1, \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2, \\
&\vdots \\
\mathbf{v}_m &= \mathbf{x}_m - \frac{\mathbf{x}_m \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_m \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \cdots - \frac{\mathbf{x}_m \cdot \mathbf{v}_{m-1}}{\mathbf{v}_{m-1} \cdot \mathbf{v}_{m-1}}\mathbf{v}_{m-1},
\end{aligned}$$

  have the following properties
  (i) $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is an orthogonal basis for $\mathcal{W}$.
  (ii) For all $k \in \{1, \ldots, m\}$ we have $\operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \operatorname{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$.
- ➢ Know the definition and how to construct a QR factorization of a matrix with linearly independent columns.

6.5 Lest square problems.

- ➢ Know the definition of a **least-squares solution** of $A\mathbf{x} = \mathbf{b}$.
- ➢ Know the theorem stating the connection between the set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$ and the set of solutions of the normal equations $A^\top A\mathbf{x} = A^\top\mathbf{b}$.
- ➢ Know the necessary and sufficient condition for the uniqueness of the least-squares solution of $A\mathbf{x} = \mathbf{b}$ (and its proof).
- ➢ Know how to find the least-squares solution of $A\mathbf{x} = \mathbf{b}$ using the QR factorization of A.
- ➢ Know how to prove the following statement: The matrices A and A^TA have the same null space.

6.6 Applications to linear models.

- ➢ Know how to find the least-squares line for a set of data points.
- ➢ Know how to find the least-squares fitting for other curves.
- ➢ Know how to find the least-squares plane for a set of data points.

7.1 Diagonalization of symmetric matrices.

- ➢ Know the theorem about the orthogonality of eigenvectors corresponding to distinct eigenvalues of a symmetric matrix and how to prove it.
- ➢ Know the definition of an orthogonally diagonalizable matrix.
- ➢ Know the relationship between the concepts of a symmetric and an orthogonally diagonalizable matrix and how to prove the easier direction of this relationship.

- ➢ Know how to prove that the eigenvalues of a symmetric 2×2 matrix are real.
- ➢ Know the spectral decomposition formula (and its meaning) for symmetric matrices: Let $A = UDU^\top$ be an orthogonal diagonalization of a symmetric matrix A. Then

$$A = UDU^\top = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \vdots \\ \mathbf{u}_n^\top \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1\mathbf{u}_1 & \lambda_2\mathbf{u}_2 & \cdots & \lambda_n\mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \vdots \\ \mathbf{u}_n^\top \end{bmatrix} = \lambda_1\mathbf{u}_1\mathbf{u}_1^\top + \lambda_2\mathbf{u}_2\mathbf{u}_2^\top + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^\top$$

(Here, for $k \in \{1, \dots, n\}$ the $n\times n$ matrix $\mathbf{u}_k\mathbf{u}_k^\top$ is the orthogonal projection matrix onto the unit vector $\mathbf{u}_k$.)

7.2 Quadratic forms.

- ➢ Know the definition of a quadratic form.
- ➢ Know how to transform a quadratic form into a quadratic form with only square terms.
- ➢ Know how to classify quadratic forms, including positive semidefinite and negative semidefinite forms.

7.3 Constrained optimization.

- ➢ Know how to solve problems like Example 3 (also including the minimum value).

7.4 The singular value decomposition.

- ➢ Know the definition of the singular value decomposition of a real $n\times m$ matrix A: A **singular value decomposition** of a real $n\times m$ matrix A is a factorization of the form

$$A = U\Sigma V^\top$$

where U is $n\times n$ orthogonal matrix, V is $m\times m$ orthogonal matrix and Σ is $n\times m$ matrix of the form

$$\Sigma = \left[\begin{array}{c|c} \begin{matrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{matrix} & 0_{r\times(m-r)} \\ \hline 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{array}\right]$$

where $r = \operatorname{rank} A$, $\begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix}$ is $r\times r$ diagonal matrix with positive entries on the diagonal and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and all the remaining entries of Σ are zeros. The values $\sigma_1, \sigma_2, \dots, \sigma_r$ are called the **singular values** of A. The columns of V are called **right singular vectors** of A and the columns of U are called **left singular vectors** of A.
- ➢ Know the consequences of the definition of a singular value decomposition. (For example, $A^\top = V\Sigma^\top U^\top$, $A^\top A = V\Sigma^\top\Sigma V^\top$, where $\Sigma^\top\Sigma$ is $m\times m$ diagonal matrix with the eigenvalues of $A^\top A$ on the diagonal and the positive entries on the diagonal are equal to $\sigma_1^2, \dots, \sigma_r^2$.)
- ➢ Know how a singular value decomposition of A contains information about orthonormal bases for all four fundamental subspaces associated with A. This is sumarized in Figure 4 on page 479.