# MATH 304 Assignment 0 September 27, 2019 Name ____________________

**Problem 1.** Let $\mathcal{V} = (-1, 1)$. Define the addition and the scalar multiplication in $\mathcal{V}$ by: For all $u, v \in \mathcal{V}$ and all $\alpha \in \mathbb{R}$ set

$$u \oplus v = \frac{u+v}{1+uv}, \qquad \alpha \odot v = \frac{(1+v)^\alpha - (1-v)^\alpha}{(1+v)^\alpha + (1-v)^\alpha}.$$

Prove that $\mathcal{V}$ with the vector addition $\oplus$ and the scaling $\odot$ is a vector space.

**Problem 2.** Consider the vector space $\mathbb{P}_2$ of all polynomials with real coefficients of degree smaller or equal than 2, defined on the real line.
We say that $p \in \mathbb{P}_2$ has a *vertex at* $t \in \mathbb{R}$ if $p(t) \leq p(x)$ for all $x \in \mathbb{R}$ or $p(t) \geq p(x)$ for all $x \in \mathbb{R}$. (This definition might be somewhat awkward since under this definition a constant polynomial has a vertex at every real number.)

(i) Let $s \in \mathbb{R}$ be an arbitrary (fixed) number. Let $\mathcal{Z}_s$ be the set all polynomials $p \in \mathbb{P}_2$ such that $p(s) = 0$, that is,

$$\mathcal{Z}_s = \{p \in \mathbb{P}_2 : p(s) = 0\}.$$

Prove that $\mathcal{Z}_s$ is a subspace of $\mathbb{P}_2$. Find a basis of this subspace. What is $\dim \mathcal{Z}_s$?

(ii) Let $t \in \mathbb{R}$ be an arbitrary (fixed) number. Let $\mathcal{V}_t$ be the set of all polynomials $p \in \mathbb{P}_2$ which have a vertex at $t$. Prove that $\mathcal{V}_t$ is a subspace of $\mathbb{P}_2$. Find a basis of this subspace. What is $\dim \mathcal{V}_t$?

(iii) Let $s, t \in \mathbb{R}$, $s \neq t$. Describe the polynomials in each of the subspaces $\mathcal{Z}_s \cap \mathcal{Z}_t$, $\mathcal{V}_t \cap \mathcal{Z}_s$ and $\mathcal{V}_s \cap \mathcal{V}_t$. Find a basis for each of these subspaces.

(iv) Let $s, t \in \mathbb{R}$ be given. Solve the equation $\mathcal{Z}_s \cap \mathcal{Z}_x = \mathcal{V}_y \cap \mathcal{Z}_t$ for $x$ and $y$.

**Problem 3.** Consider the vector space $\mathcal{V}$ of all real valued functions defined on $\mathbb{R}$, see Example 5 on page 219. The purpose of this exercise is to study some special subspaces of the vector space $\mathcal{V}$. Let $\gamma$ be an arbitrary (fixed) real number. Consider the set

$$\mathcal{S}_\gamma := \Big\{ f \in \mathcal{V} : \exists\, a, b \in \mathbb{R} \text{ such that } f(t) = a\sin(\gamma t + b) \ \ \forall\, t \in \mathbb{R} \Big\}.$$

(a) Do you see exceptional values for $\gamma$ for which the set $\mathcal{S}_\gamma$ is particularly simple?

(b) Prove that $\mathcal{S}_\gamma$ is a subspace of $\mathcal{V}$.

(c) For each $\gamma \in \mathbb{R}$ find a basis for $\mathcal{S}_\gamma$. Plot the function $\gamma \mapsto \dim \mathcal{S}_\gamma$.