Fall 2019 Math 304 Topics for the Exam 1 (version 20191016)

4.1 Vector spaces and subspaces

- Know the definition of a vector space and how to decide whether a given set is a vector space or not.
- Know the definition of a subspace of a vector space and how to decide whether a given subset of a vector space is a subspace or not.
- For given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ in a vector space $\mathcal{V}$ know the definition of a linear combination and the span, denoted by $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$.
- Know that $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is a subspace of $\mathcal{V}$.

4.2 Null spaces, column spaces, and linear transformations

- For a given $m \times n$ matrix A know the definitions of $\operatorname{Nul} A$, $\operatorname{Col} A$ and $\operatorname{Row} A$ and how to find bases for these subspaces. (See the post of October 4, Section 4.3 and Section 4.5.)

4.3 Linearly independent sets; bases

- Know the definition of linearly independent vectors and a basis for a subspace.
- Know how to decide whether given vectors from $\mathbb{R}^n$ are linearly independent or not.
- Know how to prove that the monomials $\mathbf{q}_0(t) = 1$, $\mathbf{q}_1(t) = t$, $\mathbf{q}_2(t) = t^2$, are linearly independent.
- Know how to prove that the functions $\mathbf{c}(t) = \cos t$, $\mathbf{s}(t) = \sin t$ are linearly independent.

4.4 Coordinate systems.

- Let $\mathcal{V}$ be a vector space and let $\mathcal{B}$ be a basis for $\mathcal{V}$. Know: the unique representation theorem, the definition of a coordinate mapping, the meaning of the symbol $[\mathbf{v}]_{\mathcal{B}}$ for a given vector $\mathbf{v}$ in $\mathcal{V}$.
- The importance of the matrix
$$P_{\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$$
for a given basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ for $\mathbb{R}^n$ (this is a special change-of-coordinate matrix, more in Section 4.7)
- Theorem 8: Given a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of a vector space $\mathcal{V}$, the coordinate mapping $\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$ is a one-to-one linear transformation from $\mathcal{V}$ onto $\mathbb{R}^n$. In other words, the coordinate mapping $\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$ is a linear bijection from $\mathcal{V}$ to $\mathbb{R}^n$.
- Know how a coordinate mapping for polynomials works, Examples 5 and 6
- Know Exercises 10, 11, 13

4.5 The dimension of a vector space

- The definition of a finite dimensional vector space and the definition of the dimension of a finite dimensional vector space

4.6 Rank

- Know the definition of the rank of a matrix, denoted by $\operatorname{rank} A$.
- Let m and n be positive integers. For a given $m \times n$ matrix A know the relationship between the following nonnegative integers:
$$\dim(\operatorname{Nul} A), \quad \dim(\operatorname{Col} A), \quad \dim(\operatorname{Row} A), \quad \operatorname{rank} A, \quad m, \quad n$$

4.7 Change of bases (in fact: Change of coordinates).

➢ Know that the matrix M with the property $[\mathbf{v}]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$**. It is denoted $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and it is calculated as

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \Big[\; [\mathbf{b}_1]_{\mathcal{C}} \;\cdots\; [\mathbf{b}_n]_{\mathcal{C}} \;\Big];$$

here $\mathcal{B} = \{\mathbf{b}, \ldots, \mathbf{b}_n\}$ and $\mathcal{C}$ are two bases of an n-dimensional vector space $\mathcal{V}$ and $[\mathbf{v}]_{\mathcal{B}} \in \mathbb{R}^n$ denotes the coordinate vector of $\mathbf{v} \in \mathcal{V}$ relative to the basis $\mathcal{B}$.

➢ $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$

➢ Know that there is a special basis of $\mathbb{R}^n$, called the **standard basis**, which consists of the columns of the identify matrix I_n. It is denoted by $\mathcal{E}$. This notattion comes from the fact that the vectors in this basis are commonly denoted by $\mathbf{e}_1, \cdots, \mathbf{e}_n$.

➢ Know that for a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ for $\mathbb{R}^n$ (this is a special case) we have

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \big[\, \mathbf{b}_1 \cdots \mathbf{b}_n \,\big] = P_{\mathcal{B}}$$

(the matrix $P_{\mathcal{B}}$ was introduced in Section 4.4); this basically says that $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ is a very friendly matrix.

➢ Know $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{E}}{P}\right)\left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\right) = \left(\underset{\mathcal{E}\leftarrow\mathcal{C}}{P}\right)^{-1}\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \big[\, \mathbf{c}_1 \cdots \mathbf{c}_n \,\big]^{-1}\big[\, \mathbf{b}_1 \cdots \mathbf{b}_n \,\big]$
where $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ is another basis for $\mathbb{R}^n$; this basically says that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ can be written as a product of a very friendly matrix and the inverse of another very friendly matrix

➢ Know Exercises 4 - 10, 13, 14.

5.1 Eigenvectors and eigenvalues.

➢ Know the definition of an eigenvector and eigenvalue. It is a little tricky. Pay attention.

➢ Know the definition of an eigenspace and how to find an eigenspace corresponding to a given eigenvalue.

➢ Know that the eigenvalues of a triangular matrix are the entries on its main diagonal.

➢ **Theorem.** Eigenvectors corresponding to distinct eigenvalues are linearly independent. (Or, in a more formal mathematical language: Let A be an $n{\times}n$ matrix, let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ and $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$. If $A\mathbf{v}_k = \lambda_k\mathbf{v}_k$, $\mathbf{v}_k \neq \mathbf{0}$ and $\lambda_j \neq \lambda_k$ for all $j, k = 1, 2, \ldots, m$, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent.

5.2 The characteristic equation.

➢ Know that λ is an eigenvalue of an $n{\times}n$ matrix A if and only if $\det(A-\lambda I) = 0$ (this is the characteristic equation for A)

➢ Know how to calculate $\det(A-\lambda I)$ for $2{\times}2$ and $3{\times}3$ matrices, how to find eigenvalues and corresponding eigenvectors. Exercises 1-8, but do more and find eigenvectors as well. See the post of October 10.

5.3 Diagonalization.

➢ **Theorem.** (The diagonalization theorem) An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

➢ Know how to decide whether a given $2{\times}2$ and $3{\times}3$ matrix A, is diagonalizable or not; if it is diagonalizable, how to find invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$. See the post of October 10.

➢ Know how to decide whether a triangular matrix is diagonalizable or not. Consider the matrix A in Exercise 18 in Section 5.2 and find h such that the matrix A is diagonalizable.

5.4 Eigenvectors and linear transformations.

- ➢ Know that the matrix M with the property $[T\mathbf{v}]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **matrix of T relative to the bases $\mathcal{B}$ and $\mathcal{C}$** and it is calculated as

$$M = \Big[\ [T\mathbf{b}_1]_{\mathcal{C}}\ \cdots\ [T\mathbf{b}_n]_{\mathcal{C}}\ \Big],$$

where T be a linear transformation from an m-dimensional vector space $\mathcal{V}$ to an n-dimensional vector space $\mathcal{W}$, $\mathcal{B} = \{\mathbf{b}, \dots, \mathbf{b}_m\}$ is a basis for $\mathcal{V}$ and $\mathcal{C}$ is a basis for $\mathcal{W}$.
- ➢ Know that the matrix M with the property $[T\mathbf{v}]_{\mathcal{B}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **matrix of T relative to the basis $\mathcal{B}$**, it is denoted by $[T]_{\mathcal{B}}$ and it is calculated as

$$[T]_{\mathcal{B}} = \Big[\ [T\mathbf{b}_1]_{\mathcal{B}}\ \cdots\ [T\mathbf{b}_n]_{\mathcal{B}}\ \Big],$$

where $T : \mathcal{V} \to \mathcal{V}$ is a linear transformation on an n-dimensional vector space $\mathcal{V}$ and $\mathcal{B} = \{\mathbf{b}, \dots, \mathbf{b}_n\}$ is a basis for $\mathcal{V}$.
- ➢ **Theorem.** (The diagonal matrix representation) Let A, D, P be $n\times n$ matrices, where P is invertible, D is diagonal and $A = PDP^{-1}$. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ which consists of the columns of P and if $T : \mathbb{R}^n \to \mathbb{R}^n$ is given by $T\mathbf{v} = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$, then $[T]_{\mathcal{B}} = D$.

5.5 Complex eigenvalues.

- ➢ Know how to calculate complex eigenvalues and corresponding eigenvectors of 2×2 and 3×3 real matrices
- ➢ Know that the most important class of 2×2 matrices with complex eigenvalues are rotation matrices; the matrix of the counterclockwise rotation about the origin by the angle θ measured in radians relative to the standard basis for $\mathbb{R}^2$ is

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix};$$

the 2×2 matrix R_θ is called the **rotation matrix**.
- ➢ **Theorem.** (A "hiding rotation" theorem.) Let A be a real 2×2 matrix with a nonreal eigenvalue. Then there exist an invertible 2×2 matrix P, a positive scalar α and a rotation matrix R_θ such that $A = \alpha P\, R_\theta P^{-1}$.
- ➢ In relation to the previous item see the post on October 15.

6.1 Inner product, length, and orthogonality.

- ➢ Know the definition of the dot product in $\mathbb{R}^n$, its basic properties and calculations involving it. For

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n, \qquad \mathbf{u}\cdot\mathbf{v} = \sum_{k=1}^{n} u_k\, v_k = \mathbf{u}^\top\mathbf{v} = \mathbf{v}^\top\mathbf{u}.$$

- ➢ Know the definition, the basic properties of the length of a vector in $\mathbb{R}^n$, its properties and calculations involving it.
- ➢ Know the definition of the distance in $\mathbb{R}^n$ and calculations involving it.
- ➢ Know the definition of orthogonality in $\mathbb{R}^n$ and calculations involving it.
- ➢ Know the statement and the proof of the linear algebra version of the **Pythagorean theorem**.
- ➢ Know the definition and the basic properties of the orthogonal complement in $\mathbb{R}^n$.
- ➢ Know that for a given $m \times n$ matrix A we have $(\operatorname{Row} A)^\perp = \operatorname{Nul} A$ and $(\operatorname{Col} A)^\perp = \operatorname{Nul}(A^\top)$.
- ➢ Know the geometric interpretation of the dot product in $\mathbb{R}^2$ and $\mathbb{R}^3$:

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\vartheta, \tag{1}$$

where $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^2$ or in $\mathbb{R}^3$, ϑ is the angle at the vertex O in the triangle OAB with O being the origin, A being the endpoint of $\mathbf{u}$ and B the endpoint of $\mathbf{v}$. (You should know the proof of formula (1).)