MATH 304

Assignment 1
January 24, 2020

Name ____________________

FOR FULL CREDIT JUSTIFY YOUR ANSWERS.

Problem 1. Let $\mathcal{V} = (\mathbb{R}_+)^2$. Define the addition and the scalar multiplication in $\mathcal{V}$ as follows: For all $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathcal{V}$ and all $\alpha \in \mathbb{R}$ set

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 v_1 \\ u_2 v_2 \end{bmatrix}, \qquad \alpha \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} (v_1)^\alpha \\ (v_2)^\alpha \end{bmatrix}.$$

(a) Prove that $\mathcal{V}$ with this vector addition and this vector scaling is a vector space.

(b) Describe each of the following four subspaces of $\mathcal{V}$ in the set-builder notation in terms of the coordinates of the vectors which belong to these subspaces.

$$\operatorname{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}, \quad \operatorname{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}, \quad \operatorname{Span}\left\{\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right\}, \quad \operatorname{Span}\left\{\begin{bmatrix} 4 \\ 2 \end{bmatrix}\right\}, \quad \operatorname{Span}\left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}.$$

For each of the above four subspaces provide a visual illustration in the coordinate system $\mathbb{R}_+ \times \mathbb{R}_+$.

Problem 2. Let $\mathcal{V}$ be an abstract vector space. Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathcal{V}$ are linearly independent. Prove that the vectors

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \quad \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4, \quad \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_1, \quad \mathbf{v}_4 + \mathbf{v}_1 + \mathbf{v}_2$$

are linearly independent.

Problem 3. Is the matrix A given below diagonalizable?

$$A = \begin{bmatrix} 2 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 8 & 4 & -8 & -3 \end{bmatrix}$$

Please provide all the details of your reasoning.

Problem 4. Consider the following matrix A and vector $\mathbf{v}$

$$A = \begin{bmatrix} -1 & 2 & 0 \\ -2 & 4 & 1 \\ 4 & -7 & -2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1+i \\ 1 \\ -2+i \end{bmatrix}.$$

(a) Prove that $\mathbf{v}$ is an eigenvector of A.

(b) Find an invertible real matrix P and real numbers a, b, c such that

$$P^{-1}AP = \begin{bmatrix} a & 0 & 0 \\ 0 & b & -c \\ 0 & c & b \end{bmatrix}$$

(c) Based on item (b) provide an explanation of the action of the matrix A similar to the explanation in the book on page 301.

(d) Based on item (c) state what is A^4. Explain your reasoning. No calculations are needed here.

(e) Do the items (b) and (c) above for the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Problem 5. This problem is about a specific 4×5 matrix A and its transpose. The recuced row echelon forms of each of these matrix is given below:

$$A = \begin{bmatrix} 3 & -6 & 2 & 3 & 4 \\ -3 & 6 & -2 & -3 & -4 \\ 1 & -2 & 1 & 2 & 1 \\ 3 & -6 & 1 & 0 & 5 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 2 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^\top = \begin{bmatrix} 3 & -3 & 1 & 3 \\ -6 & 6 & -2 & -6 \\ 2 & -2 & 1 & 1 \\ 3 & -3 & 2 & 0 \\ 4 & -4 & 1 & 5 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) For each of the subspaces $\operatorname{Col} A$, $\operatorname{Row} A$, $\operatorname{Nul} A$, $\operatorname{Col}(A^\top)$, $\operatorname{Row}(A^\top)$, and $\operatorname{Nul}(A^\top)$ identify the Euclidian space $\mathbb{R}^n$ of which it is a subspace. Among the six listed subspaces there are only four distinct subspaces. Identify the subspaces which are equal.

(b) Based on the RREF of A identify the following three bases. Make sure that you explain why each of the proposed sets is a basis of the corresponding subspace.

(i) The basis of the column space $\operatorname{Col} A$ of A. Call this basis $\mathcal{C}$. For each column, say $\mathbf{c}$ of A state clearly the coordinate vector $[\mathbf{c}]_{\mathcal{C}}$.

(ii) The basis of the row space $\operatorname{Row} A$ of A. Call this basis $\mathcal{B}$. For each row vector, say $\mathbf{r}$ of A state clearly the coordinate vector $[\mathbf{r}]_{\mathcal{B}}$.

(iii) The basis of the nul space $\operatorname{Nul} A$ of A. Call this basis $\mathcal{E}$.

(iv) Prove $(\operatorname{Nul} A) \cap (\operatorname{Row} A) = \{\mathbf{0}\}$. (Hint: Prove that the union of the bases $\mathcal{B}$ and $\mathcal{E}$ is a basis for the entire Euclidean space in which both of these subspaces live.)

(c) Based on the RREF of $A^\top$ identify the following three bases. Make sure that you explain why each of the proposed sets is a basis of the corresponding subspace.

(i) The basis of the column space $\operatorname{Col} A$ of A. Call this basis $\mathcal{D}$. For each column, say $\mathbf{c}$ of A state clearly the coordinate vector $[\mathbf{c}]_{\mathcal{D}}$.

(ii) The basis of the row space $\operatorname{Row} A$ of A. Call this basis $\mathcal{A}$. For each row vector, say $\mathbf{r}$ of A state clearly the coordinate vector $[\mathbf{r}]_{\mathcal{A}}$.

(iii) The basis of the nul space $\operatorname{Nul}(A^\top)$ of A. Call this basis $\mathcal{F}$.

(iv) Prove $\big(\operatorname{Nul}(A^\top)\big) \cap (\operatorname{Col} A) = \{\mathbf{0}\}$.

(d) Calculate the following four matrices:

$$\underset{\mathcal{B}\leftarrow\mathcal{A}}{P}, \quad \underset{\mathcal{A}\leftarrow\mathcal{B}}{P}, \quad \underset{\mathcal{D}\leftarrow\mathcal{C}}{P}, \quad \underset{\mathcal{C}\leftarrow\mathcal{D}}{P}.$$

(e) Do you see how two of the preceding four matrices could have been recognized immediately from the matrix A?