Winter 2020 Math 304 Topics for the Exam 1 (version 20200201)

4.1 Vector spaces and subspaces

- Know the definition of a vector space and how to decide whether a given set is a vector space or not.
- Know the definition of a subspace of a vector space and how to decide whether a given subset of a vector space is a subspace or not.
- For given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ in a vector space $\mathcal{V}$ know the definition of a linear combination and the span, denoted by $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$.
- Know that $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is a subspace of $\mathcal{V}$.

4.2 Null spaces, column spaces, and linear transformations

- For a given $m \times n$ matrix A know the definitions of $\operatorname{Nul} A$, $\operatorname{Col} A$ and $\operatorname{Row} A$ and how to find bases for these subspaces. (See the post of January 17, Section 4.3 and Section 4.5.)

4.3 Linearly independent sets; bases

- Know the definition of linearly independent vectors and a basis for a subspace.
- Know how to decide whether given vectors from $\mathbb{R}^n$ are linearly independent or not.
- Know how to prove that the monomials $\mathbf{q}_0(x) = 1$, $\mathbf{q}_1(x) = x$, $\mathbf{q}_2(x) = x^2$, are linearly independent. This proof is presented in the post on January 10.
- Know how to find a basis of a given subspace of $\mathbb{P}_2$; see examples in the post on January 10.

4.4 Coordinate systems.

- Let $\mathcal{V}$ be a vector space and let $\mathcal{B}$ be a basis for $\mathcal{V}$. Know: the unique representation theorem, the definition of a coordinate mapping, the meaning of the symbol $[\mathbf{v}]_{\mathcal{B}}$ for a given vector $\mathbf{v}$ in $\mathcal{V}$.
- Theorem 8: Given a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of a vector space $\mathcal{V}$, the coordinate mapping $\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$ is a <u>one-to-one</u> linear transformation from $\mathcal{V}$ <u>onto</u> $\mathbb{R}^n$. In other words, the coordinate mapping $\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$ is a linear bijection from $\mathcal{V}$ to $\mathbb{R}^n$.
- Know how a coordinate mapping for polynomials works, Examples 5 and 6. See also the post on January 10.
- Know Exercises 10, 11, 13

4.5 The dimension of a vector space

- The definition of a finite dimensional vector space and the definition of the dimension of a finite dimensional vector space

4.6 Rank

- Know the definition of the rank of a matrix, denoted by $\operatorname{rank} A$.
- Let m and n be positive integers. For a given $m \times n$ matrix A know the relationship between the following nonnegative integers:

$$\dim(\operatorname{Nul} A), \quad \dim(\operatorname{Col} A), \quad \dim(\operatorname{Row} A), \quad \operatorname{rank} A, \quad m, \quad n$$

4.7 Change of bases (in fact: Change of coordinates).

- ➢ Know that the matrix M with the property $[\mathbf{v}]_\mathcal{C} = M[\mathbf{v}]_\mathcal{B}$ is called the **change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$**. It is denoted $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and it is calculated as

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \Big[\ [\mathbf{b}_1]_\mathcal{C}\ \cdots\ [\mathbf{b}_n]_\mathcal{C}\ \Big];$$

here $\mathcal{B} = \{\mathbf{b}, \dots, \mathbf{b}_n\}$ and $\mathcal{C}$ are two bases of an n-dimensional vector space $\mathcal{V}$ and $[\mathbf{v}]_\mathcal{B} \in \mathbb{R}^n$ denotes the coordinate vector of $\mathbf{v} \in \mathcal{V}$ relative to the basis $\mathcal{B}$.

- ➢ $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$
- ➢ The post on February 1 has a graphical example of change-of-coordinates matrix.
- ➢ Know that there is a special basis of $\mathbb{R}^n$, called the **standard basis**, which consists of the columns of the identify matrix I_n. It is denoted by $\mathcal{E}$. This notattion comes from the fact that the vectors in this basis are commonly denoted by $\mathbf{e}_1, \cdots, \mathbf{e}_n$.
- ➢ Know that for a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ for $\mathbb{R}^n$ (this is a special case) we have

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \big[\,\mathbf{b}_1 \cdots \mathbf{b}_n\,\big] = P_\mathcal{B}$$

(the matrix $P_\mathcal{B}$ was introduced in Section 4.4); this basically says that $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ is a very friendly matrix.

- ➢ The post on January 17 has important examples of change-of-coordinates matrices in the context of the column space Col A and the row space Row A of a given matrix.
- ➢ Know Exercises 4 - 10, 13, 14.

5.1 Eigenvectors and eigenvalues.

- ➢ Know the definition of an eigenvector and eigenvalue. It is a little tricky. Pay attention.
- ➢ Know the definition of an eigenspace and how to find an eigenspace corresponding to a given eigenvalue.
- ➢ Know that the eigenvalues of a triangular matrix are the entries on its main diagonal.
- ➢ **Theorem.** Eigenvectors corresponding to distinct eigenvalues are linearly independent. (Or, in a more formal mathematical language: Let A be an $n\times n$ matrix, let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^n$ and $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$. If $A\mathbf{v}_k = \lambda_k\mathbf{v}_k$, $\mathbf{v}_k \neq \mathbf{0}$ and $\lambda_j \neq \lambda_k$ for all $j, k = 1, 2, \dots, m$, then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are linearly independent.

5.2 The characteristic equation.

- ➢ Know that λ is an eigenvalue of an $n\times n$ matrix A if and only if $\det(A-\lambda I) = 0$ (this is the characteristic equation for A)
- ➢ Know how to calculate $\det(A-\lambda I)$ for 2×2 and 3×3 matrices, how to find eigenvalues and corresponding eigenvectors. Exercises 1-8, but do more and find eigenvectors as well. See the post of January 24.

5.3 Diagonalization.

- ➢ **Theorem.** (The diagonalization theorem) An $n\times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
- ➢ Know how to decide whether a given 2×2 and 3×3 matrix A, is diagonalizable or not; if it is diagonalizable, how to find invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$. See the post of January 24.
- ➢ Know how to decide whether a triangular matrix is diagonalizable or not. Consider the matrix A in Exercise 18 in Section 5.2 and find h such that the matrix A is diagonalizable.

5.4 Eigenvectors and linear transformations.

- Know that the matrix M with the property $[T\mathbf{v}]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **matrix of T relative to the bases $\mathcal{B}$ and $\mathcal{C}$** and it is calculated as

$$M = \Big[\ [T\mathbf{b}_1]_{\mathcal{C}}\ \cdots\ [T\mathbf{b}_n]_{\mathcal{C}}\ \Big],$$

where T be a linear transformation from an m-dimensional vector space $\mathcal{V}$ to an n-dimensional vector space $\mathcal{W}$, $\mathcal{B} = \{\mathbf{b}, \dots, \mathbf{b}_m\}$ is a basis for $\mathcal{V}$ and $\mathcal{C}$ is a basis for $\mathcal{W}$.

- Know that the matrix M with the property $[T\mathbf{v}]_{\mathcal{B}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **matrix of T relative to the basis $\mathcal{B}$**, it is denoted by $[T]_{\mathcal{B}}$ and it is calculated as

$$[T]_{\mathcal{B}} = \Big[\ [T\mathbf{b}_1]_{\mathcal{B}}\ \cdots\ [T\mathbf{b}_n]_{\mathcal{B}}\ \Big],$$

where $T : \mathcal{V} \to \mathcal{V}$ is a linear transformation on an n-dimensional vector space $\mathcal{V}$ and $\mathcal{B} = \{\mathbf{b}, \dots, \mathbf{b}_n\}$ is a basis for $\mathcal{V}$.

- **Theorem.** (The diagonal matrix representation) Let A, D, P be $n\times n$ matrices, where P is invertible, D is diagonal and $A = PDP^{-1}$. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ which consists of the columns of P and if $T : \mathbb{R}^n \to \mathbb{R}^n$ is given by $T\mathbf{v} = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$, then $[T]_{\mathcal{B}} = D$.
- See the relevant examples on January 28 and January 30.

5.5 Complex eigenvalues.

- Know how to calculate complex eigenvalues and corresponding eigenvectors of 2×2 real matrices
- Know that the most important class of 2×2 matrices with complex eigenvalues are rotation matrices; the matrix of the counterclockwise rotation about the origin by the angle θ measured in radians relative to the standard basis for $\mathbb{R}^2$ is

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix};$$

the 2×2 matrix R_θ is called the **rotation matrix**.

- **Theorem.** (A "hidden rotation-dilation" theorem.) Let A be a real 2×2 matrix with a nonreal eigenvalue $a - ib$ and a corresponding eigenvector $\mathbf{u} + i\mathbf{v}$. Here $a, b \in \mathbb{R}$, $b \neq 0$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. Then the 2×2 matrix

$$P = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}$$

is invertible and

$$A = \alpha P \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} P^{-1},$$

where $\alpha = \sqrt{a^2 + b^2}$ and $\theta \in [0, 2\pi)$ is such that

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}.$$

- In relation to the previous item see the post on January 31.