Winter 2020 Math 304 Topics for the Exam 2 (version 20200301)

6.1 Inner product, length, and orthogonality.

➢ Know the definition of the dot product in $\mathbb{R}^n$, its basic properties and calculations involving it. For

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n, \qquad \mathbf{u} \cdot \mathbf{v} = \sum_{k=1}^{n} u_k\, v_k = \mathbf{u}^\top \mathbf{v} = \mathbf{v}^\top \mathbf{u}.$$

➢ Know the definition and the basic properties of the length of a vector in $\mathbb{R}^n$, its properties and calculations involving it.

➢ Know the definition of the distance in $\mathbb{R}^n$ and calculations involving it.

➢ Know the definition of orthogonality in $\mathbb{R}^n$ and calculations involving it.

➢ Know the statement and the proof of the linear algebra version of the **Pythagorean theorem**.

➢ Know the definition and the basic properties of the orthogonal complement in $\mathbb{R}^n$.

➢ Know that for a given $m \times n$ matrix A we have $(\operatorname{Row} A)^\perp = \operatorname{Nul} A$, $(\operatorname{Col} A)^\perp = \operatorname{Nul}(A^\top)$, $(\operatorname{Nul} A)^\perp = \operatorname{Row} A$, and $(\operatorname{Nul}(A^\top))^\perp = \operatorname{Col} A$.

➢ Know the geometric interpretation of the dot product in $\mathbb{R}^2$ and $\mathbb{R}^3$:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\, \|\mathbf{v}\| \cos\vartheta, \tag{1}$$

where $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^2$ or in $\mathbb{R}^3$, ϑ is the angle at the vertex O in the triangle OAB with O being the origin, A being the endpoint of $\mathbf{u}$ and B the endpoint of $\mathbf{v}$.

6.2 Orthogonal sets.

➢ Know the definition of an orthogonal set of vectors.

➢ **Theorem.** (Linear independence of orthogonal sets.) Let $\mathcal{S} = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be a subset of $\mathbb{R}^n$. If $\mathcal{S}$ is an orthogonal set which consists of nonzero vectors, then $\mathcal{S}$ is linearly independent. You should know the proof of this statement.

➢ Know the definition of an orthogonal bases.

➢ **Theorem.** (Easy expansions with orthogonal bases.) Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be an orthogonal basis of a subspace $\mathcal{W}$ of $\mathbb{R}^n$. Then for every $\mathbf{y} \in \mathcal{W}$ we have

$$\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\, \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\, \mathbf{u}_2 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_m}{\mathbf{u}_m \cdot \mathbf{u}_m}\, \mathbf{u}_m$$

➢ Know the definition of the orthogonal projection of a vector $\mathbf{y}$ onto a nonzero vector $\mathbf{u}$: A vector $\widehat{\mathbf{y}} = \alpha\mathbf{u}$ is called the **orthogonal projection of y onto u** if the difference $\mathbf{y} - \widehat{\mathbf{y}}$ is orthogonal to $\mathbf{u}$. (Convince yourself that

$$\widehat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\, \mathbf{u}$$

is the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$.)

➢ Know how to do calculations with orthogonal projections.

➢ The definitions of orthonormal set of vectors, orthonormal basis, a matrix with orthonormal columns.

➢ Know the characterization of a matrix with orthonormal columns: The columns of $n \times m$ matrix U are orthonormal if and only if $U^\top U = I_m$. (Please make sure that you understand the order of the matrix and its transpose in the previous identity.)

- ➢ Know the properties of matrices with orthonormal columns.

6.3 Orthogonal projections.

- ➢ Know the definition of the orthogonal projection of a vector $\mathbf{y}$ onto a subspace $\mathcal{W}$: A vector $\widehat{\mathbf{y}} \in \mathcal{W}$ is called the **orthogonal projection of y onto** $\mathcal{W}$ if the difference $\mathbf{y} - \widehat{\mathbf{y}}$ is orthogonal to $\mathcal{W}$. The orthogonal projection of the vector $\mathbf{y}$ onto a subspace $\mathcal{W}$ is denoted by $\operatorname{Proj}_{\mathcal{W}} \mathbf{y}$.
- ➢ **Theorem.** (The orthogonal decomposition theorem.) Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \widehat{\mathbf{y}} + \mathbf{z}$$

where $\widehat{\mathbf{y}} \in \mathcal{W}$ and $\mathbf{z} \in \mathcal{W}^{\perp}$. We have that $\widehat{\mathbf{y}} = \operatorname{Proj}_{\mathcal{W}} \mathbf{y}$. If $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is an orthogonal basis for $\mathcal{W}$, then

$$\operatorname{Proj}_{\mathcal{W}} \mathbf{y} = \widehat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_m}{\mathbf{u}_m \cdot \mathbf{u}_m} \mathbf{u}_m \tag{2}$$

- ➢ Know that equation (2) simplifies if we assume that $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is an **orthonormal basis** for $\mathcal{W}$; then

$$\operatorname{Proj}_{\mathcal{W}} \mathbf{y} = \widehat{\mathbf{y}} = (\mathbf{y} \cdot \mathbf{u}_1)\, \mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\, \mathbf{u}_2 + \cdots + (\mathbf{y} \cdot \mathbf{u}_m)\, \mathbf{u}_m. \tag{3}$$

- ➢ Know the amazing fact that equation (3) can be written as a matrix equation: Let

$$U = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_m \end{bmatrix}$$

be a matrix with **orthonormal columns**, then

$$\operatorname{Proj}_{\operatorname{Col} Q} \mathbf{y} = \widehat{\mathbf{y}} = (\mathbf{y} \cdot \mathbf{u}_1)\, \mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\, \mathbf{u}_2 + \cdots + (\mathbf{y} \cdot \mathbf{u}_m)\, \mathbf{u}_m = U\, U^{\top} \mathbf{y}.$$

The explanation for the last equality based on the definition of the matrix multiplication is as follows:

$$(\mathbf{y}\cdot\mathbf{u}_1)\, \mathbf{u}_1 + \cdots + (\mathbf{y}\cdot\mathbf{u}_m)\, \mathbf{u}_m = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_m \end{bmatrix} \begin{bmatrix} \mathbf{y} \cdot \mathbf{u}_1 \\ \vdots \\ \mathbf{y} \cdot \mathbf{u}_m \end{bmatrix} = U \begin{bmatrix} (\mathbf{u}_1)^{\top} \mathbf{y} \\ \vdots \\ (\mathbf{u}_m)^{\top} \mathbf{y} \end{bmatrix} = U \begin{bmatrix} (\mathbf{u}_1)^{\top} \\ \vdots \\ (\mathbf{u}_m)^{\top} \end{bmatrix} \mathbf{y} = U\, U^{\top} \mathbf{y}.$$

- ➢ Know how to prove the following fact: Let Q be a $n \times m$ matrix with orthonormal columns. Let $\mathbf{y} \in \mathbb{R}^n$. Prove that the projection of $\mathbf{y}$ onto the column space of Q is given by the formula $QQ^T\mathbf{y}$.
- ➢ Know how to solve Exercise 23. Given an $m \times n$ matrix A and a vector $\mathbf{v} \in \mathbb{R}^n$, know how to write $\mathbf{v}$ as a sum of a vector in $\operatorname{Nul} A$ and a vector in $\operatorname{Row} A$.
- ➢ This probelm is related to Exercse 23 in Section 6.3. Given

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 1 & 1 & 0 \\ 3 & 4 & 4 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix},$$

find a vector $\mathbf{v} \in \operatorname{Nul} A$ and a vector $\mathbf{w} \in \operatorname{Row} A$ such that

$$\mathbf{y} = \mathbf{v} + \mathbf{w}.$$

6.4 The Gram-Schmidt orthogonalization.

➢ Know the Gram-Schmidt orthogonalization process: Let m and n be positive integers such that $2 \leq m \leq n$. Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ be a basis for a subspace $\mathcal{W}$ of $\mathbb{R}^n$. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ recursively defined by

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1, \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1, \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2, \\
&\vdots \\
\mathbf{v}_m &= \mathbf{x}_m - \frac{\mathbf{x}_m \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_m \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 - \cdots - \frac{\mathbf{x}_m \cdot \mathbf{v}_{m-1}}{\mathbf{v}_{m-1} \cdot \mathbf{v}_{m-1}} \mathbf{v}_{m-1},
\end{aligned}$$

have the following properties

(i) $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is an orthogonal basis for $\mathcal{W}$.

(ii) For all $k \in \{1, \ldots, m\}$ we have $\operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \operatorname{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$.

➢ Know the definition and how to construct a QR factorization of a matrix with linearly independent columns.

6.5 Lest square problems.

➢ Know the definition of a **least-squares solution** of $A\mathbf{x} = \mathbf{b}$.

➢ Know the theorem stating the connection between the set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$ and the set of solutions of the normal equations $A^\top A\mathbf{x} = A^\top \mathbf{b}$.

➢ Know the necessary and sufficient condition for the uniqueness of the least-squares solution of $A\mathbf{x} = \mathbf{b}$ (and its proof).

➢ Know how to find the least-squares solution of $A\mathbf{x} = \mathbf{b}$ using the QR factorization of A.

➢ Know how to prove the following statement: The matrices A and $A^T A$ have the same null space.

6.6 Applications to linear models.

➢ Know how to find the least-squares line for a set of data points.

➢ Know how to find the least-squares fitting for other curves.

➢ Know how to find the least-squares plane for a set of data points.

➢ Know how to solve Exercise 14.

6.7 Inner product spaces.

➢ Know the definition of an (abstract) **inner product**.

➢ Know the definitions of length, distance and orthogonality in an inner product space.

➢ Know the statement and the proof of the abstract Pythagorean theorem.

➢ Know how to find the best approximation in an inner product space.

➢ The GramSchmidt orthogonalization algorithm in a vector space of polynomials with an inner product defined by an integral; Exercise 25 which is done in detail on the class website on February 24.

➢ Know the statement and the proof of the Cauchy-Schwarz inequality (also known as the Cauchy-BunyakovskySchwarz inequality). Know applications of the CauchyBunyakovskySchwarz inequality, like in Exercises 19 and 20 (the inequality of arithmetic and geometric means).

7.1 Diagonalization of symmetric matrices.

- ➢ Know the theorem about the orthogonality of eigenspaces corresponding to distinct eigenvalues of a symmetric matrix and how to prove it.
- ➢ Know how to prove that all the eigenvalues of a symmetric matrix are real.
- ➢ Know the definition of an orthogonally diagonalizable matrix.
- ➢ Know the relationship between the concepts of a symmetric and an orthogonally diagonalizable matrix and how to prove the easier direction of this relationship.
- ➢ Know how to prove that a symmetric 2×2 is orthogonally diagonalizable.
- ➢ Know the spectral decomposition formula (and its meaning) for symmetric matrices: Let $A = UDU^\top$ be an orthogonal diagonalization of a symmetric matrix A. Then

$$A = UDU^\top = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n\end{bmatrix} \begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix} \begin{bmatrix}\mathbf{u}_1^\top\\ \mathbf{u}_2^\top\\ \vdots\\ \mathbf{u}_n^\top\end{bmatrix}$$

$$= \begin{bmatrix}\lambda_1\mathbf{u}_1 & \lambda_2\mathbf{u}_2 & \cdots & \lambda_n\mathbf{u}_n\end{bmatrix} \begin{bmatrix}\mathbf{u}_1^\top\\ \mathbf{u}_2^\top\\ \vdots\\ \mathbf{u}_n^\top\end{bmatrix} = \lambda_1\mathbf{u}_1\mathbf{u}_1^\top + \lambda_2\mathbf{u}_2\mathbf{u}_2^\top + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^\top$$

(Here, for $k \in \{1,\ldots,n\}$ the $n\times n$ matrix $\mathbf{u}_k\mathbf{u}_k^\top$ is the orthogonal projection matrix onto the unit vector $\mathbf{u}_k$.)