Winter 2020 Math 304 Topics for the Final Exam (version 20200311)

4.1 Vector spaces and subspaces

- ➢ Know the definition of a vector space and how to decide whether a given set is a vector space or not.
- ➢ Know the definition of a subspace of a vector space and how to decide whether a given subset of a vector space is a subspace or not.
- ➢ For given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ in a vector space $\mathcal{V}$ know the definition of a linear combination and the span, denoted by $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$.
- ➢ Know that $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is a subspace of $\mathcal{V}$.

4.2 Null spaces, column spaces, and linear transformations

- ➢ For a given $m\times n$ matrix A know the definitions of $\operatorname{Nul} A$, $\operatorname{Col} A$ and $\operatorname{Row} A$ and how to find bases for these subspaces. (See the post of January 17, Section 4.3 and Section 4.5.)

4.3 Linearly independent sets; bases

- ➢ Know the definition of linearly independent vectors and a basis for a subspace.
- ➢ Know how to decide whether given vectors from $\mathbb{R}^n$ are linearly independent or not.
- ➢ Know how to prove that the monomials $\mathbf{q}_0(x) = 1$, $\mathbf{q}_1(x) = x$, $\mathbf{q}_2(x) = x^2$, are linearly independent. This proof is presented in the post on January 10.
- ➢ Know how to find a basis of a given subspace of $\mathbb{P}_2$; see examples in the post on January 10.

4.4 Coordinate systems.

- ➢ Let $\mathcal{V}$ be a vector space and let $\mathcal{B}$ be a basis for $\mathcal{V}$. Know: the unique representation theorem, the definition of a coordinate mapping, the meaning of the symbol $[\mathbf{v}]_{\mathcal{B}}$ for a given vector $\mathbf{v}$ in $\mathcal{V}$.
- ➢ Theorem 8: Given a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of a vector space $\mathcal{V}$, the coordinate mapping $\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$ is a <u>one-to-one</u> linear transformation from $\mathcal{V}$ <u>onto</u> $\mathbb{R}^n$. In other words, the coordinate mapping $\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$ is a linear bijection from $\mathcal{V}$ to $\mathbb{R}^n$.
- ➢ Know how a coordinate mapping for polynomials works, Examples 5 and 6. See also the post on January 10.
- ➢ Know Exercises 10, 11, 13

4.5 The dimension of a vector space

- ➢ The definition of a finite dimensional vector space and the definition of the dimension of a finite dimensional vector space

4.6 Rank

- ➢ Know the definition of the rank of a matrix, denoted by $\operatorname{rank} A$.
- ➢ Let m and n be positive integers. For a given $m\times n$ matrix A know the relationship between the following nonnegative integers:

$$\dim(\operatorname{Nul} A), \quad \dim(\operatorname{Col} A), \quad \dim(\operatorname{Row} A), \quad \operatorname{rank} A, \quad m, \quad n$$

4.7 Change of bases (in fact: Change of coordinates).

- ➢ Know that the matrix M with the property $[\mathbf{v}]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$**. It is denoted $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and it is calculated as

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \Big[\ [\mathbf{b}_1]_{\mathcal{C}}\ \cdots\ [\mathbf{b}_n]_{\mathcal{C}}\ \Big];$$

here $\mathcal{B} = \{\mathbf{b}, \dots, \mathbf{b}_n\}$ and $\mathcal{C}$ are two bases of an n-dimensional vector space $\mathcal{V}$ and $[\mathbf{v}]_{\mathcal{B}} \in \mathbb{R}^n$ denotes the coordinate vector of $\mathbf{v} \in \mathcal{V}$ relative to the basis $\mathcal{B}$.

- ➢ $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$
- ➢ The post on February 1 has a graphical example of change-of-coordinates matrix.
- ➢ Know that there is a special basis of $\mathbb{R}^n$, called the **standard basis**, which consists of the columns of the identify matrix I_n. It is denoted by $\mathcal{E}$. This notattion comes from the fact that the vectors in this basis are commonly denoted by $\mathbf{e}_1, \cdots, \mathbf{e}_n$.
- ➢ Know that for a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ for $\mathbb{R}^n$ (this is a special case) we have

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \big[\,\mathbf{b}_1 \cdots \mathbf{b}_n\,\big] = P_{\mathcal{B}}$$

(the matrix $P_{\mathcal{B}}$ was introduced in Section 4.4); this basically says that $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ is a very friendly matrix.

- ➢ The post on January 17 has important examples of change-of-coordinates matrices in the context of the column space $\operatorname{Col} A$ and the row space $\operatorname{Row} A$ of a given matrix.
- ➢ Know Exercises 4 - 10, 13, 14.

5.1 Eigenvectors and eigenvalues.

- ➢ Know the definition of an eigenvector and eigenvalue. It is a little tricky. Pay attention.
- ➢ Know the definition of an eigenspace and how to find an eigenspace corresponding to a given eigenvalue.
- ➢ Know that the eigenvalues of a triangular matrix are the entries on its main diagonal.
- ➢ **Theorem.** Eigenvectors corresponding to distinct eigenvalues are linearly independent. (Or, in a more formal mathematical language: Let A be an $n \times n$ matrix, let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^n$ and $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$. If $A\mathbf{v}_k = \lambda_k \mathbf{v}_k$, $\mathbf{v}_k \neq \mathbf{0}$ and $\lambda_j \neq \lambda_k$ for all $j, k = 1, 2, \dots, m$, then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are linearly independent.

5.2 The characteristic equation.

- ➢ Know that λ is an eigenvalue of an $n \times n$ matrix A if and only if $\det(A - \lambda I) = 0$ (this is the characteristic equation for A)
- ➢ Know how to calculate $\det(A - \lambda I)$ for 2×2 and 3×3 matrices, how to find eigenvalues and corresponding eigenvectors. Exercises 1-8, but do more and find eigenvectors as well. See the post of January 24.

5.3 Diagonalization.

- ➢ **Theorem.** (The diagonalization theorem) An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
- ➢ Know how to decide whether a given 2×2 and 3×3 matrix A, is diagonalizable or not; if it is diagonalizable, how to find invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$. See the post of January 24.
- ➢ Know how to decide whether a triangular matrix is diagonalizable or not. Consider the matrix A in Exercise 18 in Section 5.2 and find h such that the matrix A is diagonalizable.

5.4 Eigenvectors and linear transformations.

- Know that the matrix M with the property $[T\mathbf{v}]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **matrix of T relative to the bases $\mathcal{B}$ and $\mathcal{C}$** and it is calculated as

$$M = \Big[\, [T\mathbf{b}_1]_{\mathcal{C}} \ \cdots \ [T\mathbf{b}_n]_{\mathcal{C}} \,\Big],$$

  where T be a linear transformation from an m-dimensional vector space $\mathcal{V}$ to an n-dimensional vector space $\mathcal{W}$, $\mathcal{B} = \{\mathbf{b}, \ldots, \mathbf{b}_m\}$ is a basis for $\mathcal{V}$ and $\mathcal{C}$ is a basis for $\mathcal{W}$.
- Know that the matrix M with the property $[T\mathbf{v}]_{\mathcal{B}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **matrix of T relative to the basis $\mathcal{B}$**, it is denoted by $[T]_{\mathcal{B}}$ and it is calculated as

$$[T]_{\mathcal{B}} = \Big[\, [T\mathbf{b}_1]_{\mathcal{B}} \ \cdots \ [T\mathbf{b}_n]_{\mathcal{B}} \,\Big],$$

  where $T : \mathcal{V} \to \mathcal{V}$ is a linear transformation on an n-dimensional vector space $\mathcal{V}$ and $\mathcal{B} = \{\mathbf{b}, \ldots, \mathbf{b}_n\}$ is a basis for $\mathcal{V}$.
- **Theorem.** (The diagonal matrix representation) Let A, D, P be $n \times n$ matrices, where P is invertible, D is diagonal and $A = PDP^{-1}$. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ which consists of the columns of P and if $T : \mathbb{R}^n \to \mathbb{R}^n$ is given by $T\mathbf{v} = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$, then $[T]_{\mathcal{B}} = D$.
- See the relevant examples on January 28 and January 30.

5.5 Complex eigenvalues.

- Know how to calculate complex eigenvalues and corresponding eigenvectors of 2×2 real matrices
- Know that the most important class of 2×2 matrices with complex eigenvalues are rotation matrices; the matrix of the counterclockwise rotation about the origin by the angle θ measured in radians relative to the standard basis for $\mathbb{R}^2$ is

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix};$$

  the 2×2 matrix R_θ is called the **rotation matrix**.
- **Theorem.** (A "hidden rotation-dilation" theorem.) Let A be a real 2×2 matrix with a nonreal eigenvalue $a - ib$ and a corresponding eigenvector $\mathbf{u} + i\mathbf{v}$. Here $a, b \in \mathbb{R}$, $b \neq 0$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. Then the 2×2 matrix

$$P = \big[\mathbf{u} \ \ \mathbf{v}\big]$$

  is invertible and

$$A = \alpha P \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} P^{-1},$$

  where $\alpha = \sqrt{a^2 + b^2}$ and $\theta \in [0, 2\pi)$ is such that

$$\cos\theta = \frac{a}{\sqrt{a^2 + b^2}}, \quad \sin\theta = \frac{b}{\sqrt{a^2 + b^2}}.$$

- In relation to the previous item see the post on January 31.

6.1 Inner product, length, and orthogonality.

- Know the definition of the dot product in $\mathbb{R}^n$, its basic properties and calculations involving it. For

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n, \qquad \mathbf{u} \cdot \mathbf{v} = \sum_{k=1}^{n} u_k \, v_k = \mathbf{u}^\top \mathbf{v} = \mathbf{v}^\top \mathbf{u}.$$

- Know the definition and the basic properties of the length of a vector in $\mathbb{R}^n$, its properties and calculations involving it.
- Know the definition of the distance in $\mathbb{R}^n$ and calculations involving it.
- Know the definition of orthogonality in $\mathbb{R}^n$ and calculations involving it.
- Know the statement and the proof of the linear algebra version of the **Pythagorean theorem**.
- Know the definition and the basic properties of the orthogonal complement in $\mathbb{R}^n$.
- Know that for a given $m \times n$ matrix A we have $(\operatorname{Row} A)^\perp = \operatorname{Nul} A$, $(\operatorname{Col} A)^\perp = \operatorname{Nul}(A^\top)$, $(\operatorname{Nul} A)^\perp = \operatorname{Row} A$, and $(\operatorname{Nul}(A^\top))^\perp = \operatorname{Col} A$.
- Know the geometric interpretation of the dot product in $\mathbb{R}^2$ and $\mathbb{R}^3$:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \vartheta, \tag{1}$$

  where $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^2$ or in $\mathbb{R}^3$, ϑ is the angle at the vertex O in the triangle OAB with O being the origin, A being the endpoint of $\mathbf{u}$ and B the endpoint of $\mathbf{v}$.

6.2 Orthogonal sets.

- Know the definition of an orthogonal set of vectors.
- **Theorem.** (Linear independence of orthogonal sets.) Let $\mathcal{S} = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be a subset of $\mathbb{R}^n$. If $\mathcal{S}$ is an orthogonal set which consists of nonzero vectors, then $\mathcal{S}$ is linearly independent. You should know the proof of this statement.
- Know the definition of an orthogonal bases.
- **Theorem.** (Easy expansions with orthogonal bases.) Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be an orthogonal basis of a subspace $\mathcal{W}$ of $\mathbb{R}^n$. Then for every $\mathbf{y} \in \mathcal{W}$ we have

$$\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_m}{\mathbf{u}_m \cdot \mathbf{u}_m} \mathbf{u}_m$$

- Know the definition of the orthogonal projection of a vector $\mathbf{y}$ onto a nonzero vector $\mathbf{u}$: A vector $\widehat{\mathbf{y}} = \alpha \mathbf{u}$ is called the **orthogonal projection of y onto u** if the difference $\mathbf{y} - \widehat{\mathbf{y}}$ is orthogonal to $\mathbf{u}$. (Convince yourself that

$$\widehat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$$

  is the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$.)
- Know how to do calculations with orthogonal projections.
- The definitions of orthonormal set of vectors, orthonormal basis, a matrix with orthonormal columns.
- Know the characterization of a matrix with orthonormal columns: The columns of $n \times m$ matrix U are orthonormal if and only if $U^\top U = I_m$. (Please make sure that you understand the order of the matrix and its transpose in the previous identity.)
- Know the properties of matrices with orthonormal columns.

6.3 Orthogonal projections.

- Know the definition of the orthogonal projection of a vector $\mathbf{y}$ onto a subspace $\mathcal{W}$: A vector $\widehat{\mathbf{y}} \in \mathcal{W}$ is called the **orthogonal projection of y onto** $\mathcal{W}$ if the difference $\mathbf{y} - \widehat{\mathbf{y}}$ is orthogonal to $\mathcal{W}$. The orthogonal projection of the vector $\mathbf{y}$ onto a subspace $\mathcal{W}$ is denoted by $\operatorname{Proj}_{\mathcal{W}} \mathbf{y}$.
- **Theorem.** (The orthogonal decomposition theorem.) Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \widehat{\mathbf{y}} + \mathbf{z}$$

  where $\widehat{\mathbf{y}} \in \mathcal{W}$ and $\mathbf{z} \in \mathcal{W}^\perp$. We have that $\widehat{\mathbf{y}} = \operatorname{Proj}_{\mathcal{W}} \mathbf{y}$. If $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is an orthogonal basis for $\mathcal{W}$, then

$$\operatorname{Proj}_{\mathcal{W}} \mathbf{y} = \widehat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_m}{\mathbf{u}_m \cdot \mathbf{u}_m} \mathbf{u}_m \tag{2}$$

➢ Know that equation (2) simplifies if we assume that $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is an **orthonormal basis** for $\mathcal{W}$; then

$$\operatorname{Proj}_{\mathcal{W}} \mathbf{y} = \widehat{\mathbf{y}} = (\mathbf{y} \cdot \mathbf{u}_1)\, \mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\, \mathbf{u}_2 + \cdots + (\mathbf{y} \cdot \mathbf{u}_m)\, \mathbf{u}_m. \tag{3}$$

➢ Know the amazing fact that equation (3) can be written as a matrix equation: Let

$$U = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_m \end{bmatrix}$$

be a matrix with **orthonormal columns**, then

$$\operatorname{Proj}_{\operatorname{Col} Q} \mathbf{y} = \widehat{\mathbf{y}} = (\mathbf{y} \cdot \mathbf{u}_1)\, \mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\, \mathbf{u}_2 + \cdots + (\mathbf{y} \cdot \mathbf{u}_m)\, \mathbf{u}_m = U\, U^\top \mathbf{y}.$$

The explanation for the last equality based on the definition of the matrix multiplication is as follows:

$$(\mathbf{y} \cdot \mathbf{u}_1)\, \mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_m)\, \mathbf{u}_m = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_m \end{bmatrix} \begin{bmatrix} \mathbf{y} \cdot \mathbf{u}_1 \\ \vdots \\ \mathbf{y} \cdot \mathbf{u}_m \end{bmatrix} = U \begin{bmatrix} (\mathbf{u}_1)^\top \mathbf{y} \\ \vdots \\ (\mathbf{u}_m)^\top \mathbf{y} \end{bmatrix} = U \begin{bmatrix} (\mathbf{u}_1)^\top \\ \vdots \\ (\mathbf{u}_m)^\top \end{bmatrix} \mathbf{y} = U\, U^\top \mathbf{y}.$$

➢ Know how to prove the following fact: Let Q be a $n \times m$ matrix with orthonormal columns. Let $\mathbf{y} \in \mathbb{R}^n$. Prove that the projection of $\mathbf{y}$ onto the column space of Q is given by the formula $QQ^T \mathbf{y}$.

➢ Know how to solve Exercise 23. Given an $m \times n$ matrix A and a vector $\mathbf{v} \in \mathbb{R}^n$, know how to write $\mathbf{v}$ as a sum of a vector in $\operatorname{Nul} A$ and a vector in $\operatorname{Row} A$.

➢ This probelm is related to Exercse 23 in Section 6.3. Given

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 1 & 1 & 0 \\ 3 & 4 & 4 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix},$$

find a vector $\mathbf{v} \in \operatorname{Nul} A$ and a vector $\mathbf{w} \in \operatorname{Row} A$ such that

$$\mathbf{y} = \mathbf{v} + \mathbf{w}.$$

6.4 The Gram-Schmidt orthogonalization.

➢ Know the Gram-Schmidt orthogonalization process: Let m and n be positive integers such that $2 \leq m \leq n$. Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ be a basis for a subspace $\mathcal{W}$ of $\mathbb{R}^n$. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ recursively defined by

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1, \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1, \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2, \\
&\vdots \\
\mathbf{v}_m &= \mathbf{x}_m - \frac{\mathbf{x}_m \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_m \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 - \cdots - \frac{\mathbf{x}_m \cdot \mathbf{v}_{m-1}}{\mathbf{v}_{m-1} \cdot \mathbf{v}_{m-1}} \mathbf{v}_{m-1},
\end{aligned}$$

have the following properties

(i) $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is an orthogonal basis for $\mathcal{W}$.

(ii) For all $k \in \{1, \ldots, m\}$ we have $\operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \operatorname{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$.

- ➢ Know the definition and how to construct a QR factorization of a matrix with linearly independent columns.

6.5 Lest square problems.

- ➢ Know the definition of a **least-squares solution** of $A\mathbf{x} = \mathbf{b}$.
- ➢ Know the theorem stating the connection between the set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$ and the set of solutions of the normal equations $A^\top A\mathbf{x} = A^\top \mathbf{b}$.
- ➢ Know the necessary and sufficient condition for the uniqueness of the least-squares solution of $A\mathbf{x} = \mathbf{b}$ (and its proof).
- ➢ Know how to find the least-squares solution of $A\mathbf{x} = \mathbf{b}$ using the QR factorization of A.
- ➢ Know how to prove the following statement: The matrices A and $A^T A$ have the same null space.

6.6 Applications to linear models.

- ➢ Know how to find the least-squares line for a set of data points.
- ➢ Know how to find the least-squares fitting for other curves.
- ➢ Know how to find the least-squares plane for a set of data points.
- ➢ Know how to solve Exercise 14.

6.7 Inner product spaces.

- ➢ Know the definition of an (abstract) **inner product**.
- ➢ Know the definitions of length, distance and orthogonality in an inner product space.
- ➢ Know the statement and the proof of the abstract Pythagorean theorem.
- ➢ Know how to find the best approximation in an inner product space.
- ➢ The Gram–Schmidt orthogonalization algorithm in a vector space of polynomials with an inner product defined by an integral; Exercise 25 which is done in detail on the class website on February 24.
- ➢ Know the statement and the proof of the Cauchy-Schwarz inequality (also known as the Cauchy–Bunyakovsky-inequality). Know applications of the Cauchy–Bunyakovsky–Schwarz inequality, like in Exercises 19 and 20 (the inequality of arithmetic and geometric means).

7.1 Diagonalization of symmetric matrices.

- ➢ Know the theorem about the orthogonality of eigenspaces corresponding to distinct eigenvalues of a symmetric matrix and how to prove it.
- ➢ Know how to prove that all the eigenvalues of a symmetric matrix are real.
- ➢ Know the definition of an orthogonally diagonalizable matrix.
- ➢ Know the relationship between the concepts of a symmetric and an orthogonally diagonalizable matrix and how to prove the easier direction of this relationship.
- ➢ Know how to prove that a symmetric 2×2 is orthogonally diagonalizable.

- Know the spectral decomposition formula (and its meaning) for symmetric matrices: Let $A = UDU^\top$ be an orthogonal diagonalization of a symmetric matrix A. Then

$$A = UDU^\top = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \vdots \\ \mathbf{u}_n^\top \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1\mathbf{u}_1 & \lambda_2\mathbf{u}_2 & \cdots & \lambda_n\mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \vdots \\ \mathbf{u}_n^\top \end{bmatrix} = \lambda_1\mathbf{u}_1\mathbf{u}_1^\top + \lambda_2\mathbf{u}_2\mathbf{u}_2^\top + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^\top$$

(Here, for $k \in \{1, \ldots, n\}$ the $n \times n$ matrix $\mathbf{u}_k\mathbf{u}_k^\top$ is the orthogonal projection matrix onto the unit vector $\mathbf{u}_k$.)

7.2 Quadratic forms.

- Know the definition of a quadratic form.
- Know how to transform a quadratic form into a quadratic form with only square terms.
- Know how to classify quadratic forms, including positive semidefinite and negative semidefinite forms.

7.3 Constrained optimization.

- Know how to solve problems like Example 1 and Example 3 (also including the minimum value).

7.4 The singular value decomposition.

- Know the definition of the singular value decomposition of a real $m \times n$ matrix A: A **singular value decomposition** of a real $m \times n$ matrix A is a factorization of the form

$$A = U\Sigma V^\top$$

where U is $m \times m$ orthogonal matrix, V is $n \times n$ orthogonal matrix and Σ is $m \times n$ matrix of the form

$$\Sigma = \left[\begin{array}{ccc|c} \sigma_1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & 0_{r \times (n-r)} \\ 0 & \cdots & \sigma_r & \\ \hline & 0_{(m-r) \times r} & & 0_{(m-r) \times (n-r)} \end{array}\right]$$

where $r = \operatorname{rank} A$, $\begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix}$ is $r \times r$ diagonal matrix with positive entries on the diagonal and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and all the remaining entries of Σ are zeros. The values $\sigma_1, \sigma_2, \ldots, \sigma_r$ are called the **singular values** of A. The columns of V are called **right singular vectors** of A and the columns of U are called **left singular vectors** of A.

- Know the consequences of the definition of a singular value decomposition. (For example, $A^\top = V\Sigma^\top U^\top$, $A^\top A = V\Sigma^\top\Sigma V^\top$, where $\Sigma^\top\Sigma$ is $n \times n$ diagonal matrix with the eigenvalues of $A^\top A$ on the diagonal and the positive entries on the diagonal are equal to $\sigma_1^2, \ldots, \sigma_r^2$.)

- Know how a singular value decomposition of A contains orthonormal bases for all four fundamental subspaces associated with A. This is summarized in Figure 4 on page 423:

  - the columns of V form an orthonormal basis for $\mathbb{R}^n$,
  - the first r columns of V form an orthonormal basis for $\operatorname{Row} A$,
  - the last $n-r$ columns of V form an orthonormal basis for $\operatorname{Nul} A$,
  - the columns of U form an orthonormal basis for $\mathbb{R}^m$,
  - the first r columns of U form an orthonormal basis for $\operatorname{Col} A$,
  - the last $m-r$ columns of U form an orthonormal basis for $\operatorname{Nul}(A^\top)$.

- Review the singular value decomposition of the matrix we found on Wikipedia which we did in our first online class; see the post on March 11.