An almost magic proposition about infinite subsets of $\mathbb{R}$

The proposition below is an implication of the form: $P \Rightarrow Q \vee R$. This implication is equivalent to the implication $P \wedge \neg Q \Rightarrow R$. One way to see this is to consider the negations of these implications. The negation of $P \Rightarrow Q \vee R$ is $P \wedge (\neg Q \wedge \neg R)$, while the negation of $P \wedge \neg Q \Rightarrow R$ is $(P \wedge \neg Q) \wedge \neg R$. Since the negations are clearly equivalent, the implications are also equivalent.

Proposition. *Let $A \subset \mathbb{R}$. If A is infinite, then there exists a nonempty subset B of A such that B does not have a minimum or there exists a nonempty subset C of A such that C does not have a maximum.*

Proof. We will prove the equivalent implication: If A is an infinite subset of $\mathbb{R}$ and each nonempty subset of A has a minimum, then there exist a nonempty subset C of A such that C does not have a maximum.

So, assume that A is an infinite subset of $\mathbb{R}$ and each nonempty subset of A has a minimum. Then, in particular, $\min A$ exists. Let W be the set of all minimums of infinite subsets of A. Formally,

$$W = \Bigl\{ x \in A \ : \ x = \min E \ \text{ where } \ E \subset A \ \text{ and } \ E \text{ is infinite} \Bigr\}.$$

Clearly $\min A$ is an element in W. Hence $W \neq \emptyset$.

Next we will prove that W does not have a maximum. Let $y \in W$ be arbitrary. Then there exists an infinite subset F of A such that $y = \min F$. Since F is infinite, the set $F \setminus \{y\}$ is also infinite. Since $F \setminus \{y\} \subset A$, by the assumption $z = \min\bigl(F \setminus \{y\}\bigr)$ exists. Therefore, $z \in W$. Since $z \in F \setminus \{y\}$, we have $z \neq y$. Since $z \in F$ and $y = \min F$, we have $z \geq y$. Hence $z > y$. Thus, for each $y \in W$ there exists $z \in W$ such that $z > y$. This proves that W is a nonempty subset of A which does not have a maximum. $\square$