Axiom 17 (New Completeness Axiom)**.** Let A and B be nonempty subsets of $\mathbb{R}$. If $x \le y$ for all $x \in A$ and all $y \in B$, then there exists $c \in \mathbb{R}$ such that $x \le c \le y$ for all $x \in A$ and all $y \in B$.

Exercise 1. Let $n \in \mathbb{N}$ and let a be a positive real number. Prove that there exists a positive real number α such that $\alpha^n = a$.

Solution. Notice that the statement is trivial for $a = 1$. Then, clearly $\alpha = 1$. Therefore, in the rest of the proof we assume that $a > 0$ and $a \neq 1$.

Next we define the sets A and B:

$$A = \{x \in \mathbb{R} : x > 0 \ \text{ and } \ x^n \le a\}$$

and

$$B = \{y \in \mathbb{R} : y > 0 \ \text{ and } \ y^n \ge a\}.$$

These sets have the following three properties.

1. $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$. (This property is clear.)

2. The sets A and B are nonempty sets.

To prove this property we consider two cases for a:

CASE 1. Assume $a < 1$. Then it can be proved by induction that $a^n > 0$ and $a^n \le a$. Therefore, $a \in A$. Also, by induction $a < 1^n$. Therefore $1 \in B$.

CASE 2. Assume $a > 1$. Then, by induction, $a^n \ge a > 0$. Therefore, $a \in B$. Also $1^n < a$. Thus $1 \in A$.

3. For all $x \in A$ and for all $y \in B$, $x \le y$.

To prove this property assume $x \in A$ and $y \in B$. Then, $x > 0$ and $y > 0$ and $x^n \le a \le y^n$. Therefore, $x^n \le y^n$. By Exercise 2.7.3, this implies $x \le y$.

Now we can apply the New Completeness Axiom to the sets A and B and conclude that there exists $c \in \mathbb{R}$ such that

$$x \le c \le y \quad \text{for all} \quad x \in A \quad \text{and} \quad \text{for all} \quad y \in B. \tag{1}$$

Notice that (1) and the definition of A imply $c > 0$.

Next we will prove two implications:

$$x \le c \ \text{ for all } \ x \in A \quad \Rightarrow \quad c^n \ge a, \tag{2}$$

and

$$c \le y \ \text{ for all } \ y \in B \quad \Rightarrow \quad c^n \le a. \tag{3}$$

Relation (1) and implications (2) and (3) yield $c^n = a$.

PROOF OF IMPLICATION (2). We will prove the contrapositive. Assume $s > 0$. Then

$$s^n < a \quad \Rightarrow \quad \exists\, u \in A \quad \text{such that} \quad u > s.$$

So, assume $s^n < a$. Notice that $s^n + (n-1)a > 0$ and set

$$u = \frac{s\,a\,n}{s^n + (n-1)a}.$$

Since $s > 0$ and $s^n < a$ we have $0 < s^n + (n-1)a < a + (n-1)a = n\,a$. Therefore

$$u = \frac{s\,a\,n}{s^n + (n-1)a} > \frac{s\,a\,n}{n\,a} = s.$$

Hence $u > s$. The question now is whether $u \in A$. Since $u > 0$ to prove $u \in A$ we need to prove $u^n \leq a$. To prove this inequality we will use Bernoulli's inequality: If $x > -1$ and $n \in \mathbb{N}$, then

$$(1+x)^n \geq 1 + n\,x.$$

Since

$$0 > \left(\frac{s^n}{a} - 1\right)\frac{1}{n} > -\frac{1}{n} > -1,$$

Bernoulli's inequality implies

$$\left(1 + \left(\frac{s^n}{a} - 1\right)\frac{1}{n}\right)^n \geq 1 + n\left(\frac{s^n}{a} - 1\right)\frac{1}{n} = \frac{s^n}{a}.$$

We are now ready to prove that $u^n \leq a$:

$$u^n = \left(\frac{s\,a\,n}{s^n + (n-1)a}\right)^n = \frac{s^n}{\left(\frac{s^n}{na} + 1 - \frac{1}{n}\right)^n} = \frac{s^n}{\left(1 + \left(\frac{s^n}{a} - 1\right)\frac{1}{n}\right)^n} \leq \frac{s^n}{\frac{s^n}{a}} = a$$

Thus we proved that $u \in A$ and $s < u$. This completes the proof of the contrapositive of implication (2).

Proof of implication (3). We will prove the contrapositive. Assume $t > 0$. Then

$$t^n > a \quad \Rightarrow \quad \exists\, v \in B \quad \text{such that} \quad v < t.$$

So, assume $t^n > a$. Set

$$v = \frac{n-1}{n}\,t + \frac{a}{n\,t^{n-1}}\,.$$

Since $t^n > a$, $a/(t^n) < 1$. Therefore,

$$v = \frac{n-1}{n}\,t + \frac{a}{n\,t^{n-1}} = t\left(1 - \frac{1}{n} + \frac{1}{n}\frac{a}{t^n}\right) < t\left(1 - \frac{1}{n} + \frac{1}{n}\right) = t.$$

Hence $v < t$. Clearly $v > 0$. Next we prove $v^n \geq a$. Since

$$0 > \left(\frac{a}{t^n} - 1\right)\frac{1}{n} > -\frac{1}{n} > -1,$$

we can use Bernoulli's inequality again:

$$\begin{aligned}
v^n &= \left(\frac{n-1}{n}\,t + \frac{a}{n\,t^{n-1}}\right)^n \\
&= t^n\left(1 - \frac{1}{n} + \frac{a}{n\,t^n}\right)^n \\
&= t^n\left(1 + \left(\frac{a}{t^n} - 1\right)\frac{1}{n}\right)^n \\
&\geq t^n\left(1 + n\left(\frac{a}{t^n} - 1\right)\frac{1}{n}\right) \\
&= t^n\left(1 + \frac{a}{t^n} - 1\right) \\
&= a
\end{aligned}$$

Thus $v \in B$. Since we already proved $v < t$, the contrapositive of implication (3) is proved.

This completes the solution of the exercise. □