MATH 312 Examination 2 November 24, 2009

Name Key

Problem 1. Prove that the set of all functions $f : \mathbb{N} \to \{0,1\}$ is not countable. (This set of functions is denoted by $\{0,1\}^{\mathbb{N}}$.)

Problem 2. (a) Prove that the set $\mathbb{N}$ is not bounded.

(b) Let a an b be real numbers such that $a < b$. Prove that there exist $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that

$$a < \frac{m}{n} < b.$$

Problem 3. Prove that there exists a positive real number α such that $\alpha^2 = 2$.

Problem 4. (a) Let $\{s_n\}$ be a non-decreasing sequence which is bounded above. Prove that the sequence $\{s_n\}$ converges.

(b) Let A be a nonempty and bounded above subset of $\mathbb{R}$. Set $a = \sup A$. Prove that there exists a sequence $\{x_n\}$ with the following properties:

- $x_n \in A$ for all $n \in \mathbb{N}$.
- $\lim_{n\to\infty} x_n = a$.

① Set $\mathcal{F} = \{0,1\}^{\mathbb{N}}$. $\boxed{1}$

Let $\Phi : \mathbb{N} \to \mathcal{F}$ be an arbitrary function. We will prove that Φ is not a surjection by constructing $f \in \mathcal{F}$ such that $f \neq \Phi_n$ $\forall n \in \mathbb{N}$.

Simply set

$$f(n) = 1 - \Phi_n(n), \quad \forall n \in \mathbb{N}.$$

Since $\Phi_n(n) \in \{0,1\}$, $1 - \Phi_n(n) \in \{0,1\}$.

Hence $f \in \mathcal{F}$. Since

$$f(n) \neq \Phi_n(n)$$

we have $f \neq \Phi_n$, and this holds for all $n \in \mathbb{N}$. Hence f is not in the range of Φ. So Φ is not a surjection.

Consequently $\mathcal{F}$ is not countable.

2 (a) A direct proof.

First a lemma.

<u>Lemma.</u> Let A be a nonempty set which does not have a maximum. The following implication holds:

$$(*) \quad A \text{ bdd above} \Rightarrow \forall \varepsilon>0\ \exists x,y\in A \text{ s.t. } x<y \text{ and } y-x<\varepsilon$$

<u>Proof.</u> Assume A is bdd above. Since $A\neq\emptyset$ $\sup A = a$ exists. Since A d.n. have max $a\notin A$. By Ex... $\forall \varepsilon>0\ \exists x\in A$ such that $a-\varepsilon<x<a$. Here $x<a$ since $x\in A$ and $a\notin A$ and $x\leq a$. Since $a-x>0$, by the same Exercise $\exists y\in A$ such that

$$\underbrace{a-(a-x)}_{=x}<y<a.$$

Thus

$$a-\varepsilon<x<y<a$$

Therefore $y-x<a-(a-\varepsilon)=\varepsilon$.

This proves $(*)$.

The CP of $(*)$ is

$\exists \varepsilon>0$ s.t. $\forall x,y\in A$ s.t. $x<y$ we have $y-x\geq\varepsilon$

$\Rightarrow A$ not bdd above.

Since we proved that

$n, m \in \mathbb{N}$ and $n > m \Rightarrow n - m \geq 1$ $\boxed{3}$

(must be somewhere in the notes)

the contrapositive of the Lemma yields that $\mathbb{N}$ is <u>not</u> bdd above.

(b) Let $n \in \mathbb{N}$ be such that

$$\frac{1}{n} < b - a$$

(Such n exists since $b - a > 0$).

Then $na + 1 < nb$. We will use the following properties of $\lceil u \rceil$: $u \leq \lceil u \rceil < u + 1$

Thus $\lceil na \rceil < na + 1 < nb$

Also $na \leq \lceil na \rceil < \lceil na \rceil + 1$.

thus $na < \lceil na \rceil + 1 < nb$

and $a < \dfrac{\lceil na \rceil + 1}{n} < b$.

Since $\lceil na \rceil + 1 \in \mathbb{Z}$ and $n \in \mathbb{N}$,

(b) is proved.

The end of the proof of Problem 2(b) is wrong. Here is a correct proof.

Since $\mathbb{N}$ is not bounded above there exists $n \in \mathbb{N}$ such that $\dfrac{1}{b-a} < n$. Since $b - a > 0$ we then have $1 < nb - na$. That is $na < nb - 1$. We will use the following property of the ceiling function:

$$u \leq \lceil u \rceil < u + 1 \quad \text{for all} \quad x \in \mathbb{R}.$$

Applied to $u = nb$ we get

$$nb \leq \lceil nb \rceil < nb + 1,$$

or

$$nb - 1 \leq \lceil nb \rceil - 1 < nb.$$

Since $na < nb - 1$, we have

$$na < nb - 1 \leq \lceil nb \rceil - 1 < nb,$$

and consequently

$$a < \frac{\lceil nb \rceil - 1}{n} < b.$$

Since $\lceil nb \rceil - 1 \in \mathbb{Z}$ and $n \in \mathbb{N}$ the proof is complete.

③ Set

$$A = \{x \in \mathbb{R} : x > 0 \text{ and } x^2 < 2\}$$
$$B = \{y \in \mathbb{R} : y > 0, \text{ and } y^2 > 2\}.$$

$\boxed{4}$

Since $1 \in A$ and $2 \in B$, $A \neq \emptyset$ and $B \neq$
$\forall x \in A\ \forall y \in B$ we have $x^2 < y^2$ and thus $x < y$. By CA $\exists\ \alpha \in \mathbb{R}$ such that

$$x \leq \alpha \leq y \quad \forall x \in A\ \forall y \in B.$$

Since $x > 0\ \forall x \in A$ we have $\alpha > 0$.
We will prove later that A has no max and B has no min. ~~Hence~~ Since α is a lower bound for B, $\alpha \notin B$ and since α is an upper bound for A, $\alpha \notin A$.
Thus, $\alpha^2 \geq 2$ (since $\alpha \notin B$) and

$$\alpha^2 \leq \alpha \ (\text{since } \alpha \notin A)$$

Hence $\alpha^2 = 2$.

Note that $(s+t)^2 > 4st$ whenever $s, t > 0$, $s \neq t$

Proof of B has no ~~max~~ min.
Let $y \in B$. Then $\frac{y}{2} + \frac{1}{y} \in B$

Hence $\left(\frac{s}{2} + \frac{1}{s}\right)^2 > 2$ whenever $s^2 \neq 2$

and $y > \frac{y}{2} + \frac{1}{y}$ (Since

$y^2 > 2 \Rightarrow \frac{y}{2} > \frac{1}{y} \Rightarrow y > \frac{y}{2} + 1$

A proof for

4. (a) Consider the set $\boxed{5}$

$$A = \{ s_n : n \in \mathbb{N} \}$$

$A \neq \emptyset$ since $s_1 \in A$ and A is bdd above

therefore $a = \sup A$ exists.

We will prove that $\lim_{n \to \infty} s_n = a$.

Let $\varepsilon > 0$ be arbitrary. Then by Ex...

$\exists\, x_\varepsilon \in A$ such that

$$a - \varepsilon < x_\varepsilon \leq a.$$

But $x_\varepsilon \in \{ s_n : n \in \mathbb{N} \}$. Hence $\exists\, n_\varepsilon \in \mathbb{N}$

such that $x_\varepsilon = s_{n_\varepsilon}$. But $\{s_n\}$

is nondecreasing, so

$$s_{n_\varepsilon} \leq s_n \text{ for all } n \in \mathbb{N},\ n \geq n_\varepsilon.$$

Therefore

$\forall n \in \mathbb{N},\ n \geq n_\varepsilon \Rightarrow |s_n - a| = a - s_n \leq a - s_{n_\varepsilon} =$

$$= a - x_\varepsilon < a - (a - \varepsilon) = \varepsilon$$

this proves that $\lim_{n \to \infty} s_n = a$.

4. (b) By Ex... $\boxed{6}$

$\forall \varepsilon > 0 \; \exists y(\varepsilon) \in A$ s.t.

$$a - \varepsilon < y(\varepsilon) \leq a.$$

Let $n \in \mathbb{N}$ and set

$$x_n = y(1/n).$$

Then $x_n \in A \; \forall n \in \mathbb{N}$.

Let $\varepsilon > 0$. Set $N(\varepsilon) = 1/\varepsilon$.

Let $n \in \mathbb{N}$, $n > 1/\varepsilon$. Then $\frac{1}{n} < \varepsilon$

and

$$|x_n - a| = a - x_n = a - y(1/n) < \frac{1}{n} < \varepsilon.$$

This proves that $\lim_{n \to \infty} x_n = a$.