ON THE MAXIMUM OF A CONTINUOUS FUNCTION

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Theorem. *Let $a, b \in \mathbb{R}$, $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Then there exists $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$.*

Proof. Case I. Assume $f(a) \geq f(x)$ for all $x \in [a, b]$. Then we can take $c = a$.

Case II. Assume that there exists $s \in [a, b]$ such that $f(s) > f(a)$. Set

$$W = \Big\{ w \in [a, b) \,:\, \exists\, z \in [a, b] \ \text{ such that } \ f(x) < f(z) \ \ \forall\, x \in [a, w] \ \Big\}.$$

Step 1. In this case we have $a \in W$. Just set $z = s$ and $f(x) < f(z)$ is true for all $x \in [a, a] = \{a\}$. By definition, $W \subset [a, b)$. Therefore

$$c = \sup W$$

exists by the Completeness Axiom. Clearly $c \in [a, b]$.

Step 2. Here we show that W does not have a maximum. Let $v \in W$ be arbitrary. Then $v < b$ and there exists $z \in [a, b]$ such that

$$f(x) < f(z) \quad \forall\, x \in [a, v]. \tag{1}$$

In particular, $f(v) < f(z)$. Set $\epsilon_0 = \frac{1}{2}\big(f(z) - f(v)\big) > 0$. Since f is continuous at v, there exists $\delta_0 = \delta_v(\epsilon_0) > 0$ such that

$$x \in [a, b] \cap \big(v - \delta_0, v + \delta_0\big) \quad \Rightarrow \quad f(v) - \epsilon_0 < f(x) < f(v) + \epsilon_0. \tag{2}$$

Set $\mu = \frac{1}{2} \min\big\{\delta_0, b - v\big\} > 0$. Then $v + \mu < b$ and $v + \mu < v + \delta_0$. Now (2) implies

$$f(x) < f(v) + \epsilon_0 = \frac{1}{2}\big(f(v) + f(z)\big) < f(z) \quad \forall\, x \in [v, v + \mu]. \tag{3}$$

It follows from (1) and (3) that

$$f(x) < f(z) \quad \forall\, x \in [a, v + \mu].$$

Consequently $v + \mu \in W$. Hence v is not a maximum of W. Thus, $c \notin W$. In particular $c > a$.

Step 3. Here we show that $[a, c) \subset W$. Let $x \in [a, c)$ be arbitrary. Since $x < c$ and $c = \sup W$, x is not an upper bound of W. Hence, there exists $w \in W$ such that $x < w < c$. Now $x \in W$ follows directly from the definition of W. Thus $[a, c) \subset W$.

Step 4. Next we prove the implication:

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$$a < c \text{ and } [a,c) \subset W \text{ and } c \notin W \implies f(c) \geq f(x) \ \ \forall\, x \in [a,b].$$

The following implication is a partial contrapositive of the preceding one and hence equivalent to it:

$$a < v \text{ and } [a,v) \subset W \text{ and } \exists\, t \in [a,b] \text{ s.t. } f(t) > f(v) \implies v \in W.$$

Since this implication is easier to prove, we proceed with its proof in the next step.

Step 5. Assume $a < v$, $[a,v) \subset W$ and let $t \in [a,b]$ be such that $f(t) > f(v)$. Set $\epsilon_1 = \frac{1}{2}\bigl(f(t) - f(v)\bigr) > 0$. Since f is continuous at v there exists $\delta_1 = \delta_v(\epsilon_1) > 0$ such that

$$x \in [a,b] \cap \bigl(v - \delta_1, v + \delta_1\bigr) \quad \Rightarrow \quad f(v) - \epsilon_1 < f(x) < f(v) + \epsilon_1. \tag{4}$$

Now set $\eta = \frac{1}{2}\min\bigl\{\delta_1, v - a\bigr\} > 0$. Then $a < v - \eta$ and $v - \delta_1 < v - \eta$. Therefore, by (4) we have

$$f(x) < f(v) + \epsilon_1 = \frac{1}{2}\bigl(f(v) + f(t)\bigr) < f(t) \quad \forall\, x \in [v - \eta, v].$$

Or, briefly,

$$f(x) < f(t) \quad \forall\, x \in [v - \eta, v]. \tag{5}$$

Since $a < v - \eta < v$, the assumption $[a,v) \subset W$ gives $v - \eta \in W$. Therefore, there exists $s \in [a,b]$ such that

$$f(x) < f(s) \quad \forall\, x \in [a, v - \eta]. \tag{6}$$

To prove that $v \in W$, we set

$$z = \begin{cases} s & \text{if } \ f(t) < f(s), \\ t & \text{if } \ f(s) \leq f(t). \end{cases}$$

Then, clearly,

$$f(z) = \max\bigl\{f(t), f(s)\bigr\}.$$

Therefore, (5) and (6) imply

$$f(x) < f(z) \quad \forall\, x \in [a, v].$$

Thus, $v \in W$.

Conclusion. The second implication in Step 4 is proved in Step 5. Since two implications in Step 4 are equivalent, we have proved the first implication in Step 4. Since the hypotheses of the first implication in Step 4 are true by Steps 2 and 3, we have proved that $f(c) \geq f(x)$ for all $x \in [a,b]$. The proof is complete. □