CHAPTER 3

Sequences in $\mathbb{R}$

3.1. Definitions and examples

Definition 3.1.1. A *sequence in* $\mathbb{R}$ is a function whose domain is $\mathbb{N}$ and whose range is in $\mathbb{R}$.

Let $s : \mathbb{N} \to \mathbb{R}$ be a sequence in $\mathbb{R}$. Then the values of s are

$$s(1),\ s(2),\ s(3),\ \ldots,\ s(n),\ \ldots\ .$$

It is customary to write s_n instead of $s(n)$ for the values of a sequence. Sometimes a sequence will be specified by listing its first few terms

$$s_1,\ s_2,\ s_3,\ s_4,\ \ldots\ ,$$

and sometimes by listing of all its terms $\{s_n\}_{n=1}^{\infty}$ or $\{s_n\}$. One way of specifying a sequence is to give a formula, or a recursion formula for its $n-$th term s_n.

Remark 3.1.2. In the above notation s is the "name" of the sequence and $n \in \mathbb{N}$ is the independent variable.

Remark 3.1.3. Notice the difference between the following two expressions:

$\{s_n\}_{n=1}^{\infty}$ This expression denotes a function (sequence).

$\{s_n\ :\ n \in \mathbb{N}\}$ This expression denotes a set: The range of a sequence $\{s_n\}_{n=1}^{\infty}$.

For example $\{1-(-1)^n\}_{n=1}^{\infty}$ stands for the function $n \mapsto 1-(-1)^n$, $n \in \mathbb{N}$, while

$$\{1-(-1)^n : n \in \mathbb{N}\} = \{0, 2\}.$$

Example 3.1.4. Here we give examples of sequences given by a formula. In each formula below $n \in \mathbb{N}$.

$$\begin{array}{llllllll}
\text{(a)} & n, & \text{(b)} & n^2, & \text{(c)} & \sqrt{n}, & \text{(d)} & (-1)^n, \\[2ex]
\text{(e)} & \dfrac{1}{n}, & \text{(f)} & \dfrac{1}{n^2}, & \text{(g)} & \dfrac{1}{\sqrt{n}}, & \text{(h)} & 1-\dfrac{(-1)^n}{n}, \\[2ex]
\text{(i)} & \dfrac{1}{n!}, & \text{(j)} & 2^{1/n}, & \text{(k)} & n^{1/n}, & \text{(l)} & n^{(-1)^n}, \\[2ex]
\text{(m)} & \dfrac{9^n}{n!}, & \text{(n)} & \dfrac{(-1)^{n+1}}{2n-1}, & \text{(o)} & \dfrac{n^{(-1)^n}}{n+1}, & \text{(p)} & \left(\dfrac{e}{n}\right)^n \dfrac{n!}{\sqrt{n}}.
\end{array}$$

Example 3.1.5. Few more sequences given by a formula are

$$\text{(a)}\ \left\{\sqrt{n^2+1}-n\right\}_{n=1}^{\infty},\quad \text{(b)}\ \left\{\sqrt{n^2+n}-n\right\}_{n=1}^{\infty},\quad \text{(c)}\ \left\{\sqrt{n+1}-\sqrt{n}\right\}_{n=1}^{\infty}.$$

Example 3.1.6. In this example we give several recursively defined sequences.

(a) $s_1 = 1$ and $(\forall n \in \mathbb{N})$ $s_{n+1} = -\dfrac{s_n}{2}$,

(b) $x_1 = 1$ and $(\forall n \in \mathbb{N})$ $x_{n+1} = 1 + \dfrac{x_n}{4}$,

(c) $x_1 = 2$ and $(\forall n \in \mathbb{N})$ $x_{n+1} = \dfrac{x_n}{2} + \dfrac{1}{x_n}$,

(d) $a_1 = \sqrt{2}$ and $(\forall n \in \mathbb{N})$ $a_{n+1} = \sqrt{2 + a_n}$,

(e) $s_1 = 1$ and $(\forall n \in \mathbb{N})$ $s_{n+1} = \sqrt{1 + s_n}$,

(f) $x_1 = 0$ and $(\forall n \in \mathbb{N})$ $x_{n+1} = \dfrac{9 + x_n}{10}$.

For a recursively defined sequence it is useful to evaluate the values of the first few terms to get an idea how sequence behaves.

Example 3.1.7. The most important examples of sequences are listed below:

$$b_n = a, \; n \in \mathbb{N}, \quad \text{where} \quad a \in \mathbb{R}, \tag{3.1.1}$$

$$p_n = a^n, \; n \in \mathbb{N}, \quad \text{where} \quad -1 < a < 1, \tag{3.1.2}$$

$$E_n = \left(1 + \frac{1}{n}\right)^n, \quad n \in \mathbb{N}, \tag{3.1.3}$$

$$G_1 = a + ax \;\text{ and }\; (\forall\, n \in \mathbb{N})\; G_{n+1} = G_n + ax^{n+1}, \;\text{ where }\; -1 < x < 1, \tag{3.1.4}$$

$$S_1 = 2 \;\text{ and }\; (\forall\, n \in \mathbb{N})\; S_{n+1} = S_n + \frac{1}{(n+1)!}, \tag{3.1.5}$$

$$v_1 = 1 + a \;\text{ and }\; (\forall\, n \in \mathbb{N})\; v_{n+1} = v_n + \frac{a^{n+1}}{(n+1)!}, \;\text{ where }\; a \in \mathbb{R}. \tag{3.1.6}$$

Definition 3.1.8. Let $\{a_n\}$ be a sequence in $\mathbb{R}$. A sequence which is recursively defined by

$$S_1 = a_1 \;\text{ and }\; (\forall\, n \in \mathbb{N})\; S_{n+1} = S_n + a_{n+1}, \tag{3.1.7}$$

is called a *sequence of partial sum* corresponding to $\{a_n\}$.

Example 3.1.9. The sequences of partial sums associated with the sequences in Example 3.1.4 (e), (f) and (n) are important examples for Definition 3.1.8. Notice also that the sequences in (3.1.4), (3.1.5) and (3.1.6) are sequences of partial sums. All of these are very important.

3.2. Bounded sequences

Definition 3.2.1. Let $\{s_n\}$ be a sequence in $\mathbb{R}$.

(1) If a real number M satisfies

$$s_n \le M \quad \text{for all} \quad n \in \mathbb{N}$$

then M is called an *upper bound* of $\{s_n\}$ and the sequence $\{s_n\}$ is said to be *bounded above*.

(2) If a real number m satisfies

$$m \le s_n \quad \text{for all} \quad n \in \mathbb{N},$$

then m is called a *lower bound* of $\{s_n\}$ and the sequence $\{s_n\}$ is said to be *bounded below*.

(3) The sequence $\{s_n\}$ is said to be *bounded* if it is bounded above and bounded below.

Remark 3.2.2. Clearly, a sequence $\{s_n\}$ is bounded above if and only if the set $\{s_n : n \in \mathbb{N}\}$ is bounded above. Similarly, a sequence $\{s_n\}$ is bounded below if and only if the set $\{s_n : n \in \mathbb{N}\}$ is bounded below.

Remark 3.2.3. The sequence $\{s_n\}$ is bounded if and only if there exists a real number $K > 0$ such that $|s_n| \leq K$ for all $n \in \mathbb{N}$.

Exercise 3.2.4. There is a huge task here. For each sequence given in this section it is of interest to determine whether it is bounded or not. As usual, some of the proofs are easy, some are hard. It is important to do few easy proofs and observe their structure. This will provide the setting to appreciate proofs for hard examples.

3.3. The definition of a convergent sequence

Definition 3.3.1. A sequence $\{s_n\}$ is a *constant* sequence if there exists $L \in \mathbb{R}$ such that $s_n = L$ for all $n \in \mathbb{N}$.

Exercise 3.3.2. Prove that the sequence $s_n = \left\lfloor \dfrac{3n-1}{2n} \right\rfloor$, $n \in \mathbb{N}$, is a constant sequence.

Definition 3.3.3. A sequence $\{s_n\}$ is *eventually constant* if there exists $L \in \mathbb{R}$ and $N_0 \in \mathbb{R}$ such that $s_n = L$ for all $n \in \mathbb{N}$, $n > N_0$.

Exercise 3.3.4. Prove that the sequence $s_n = \left\lceil \dfrac{3n-2}{2n+3} \right\rceil$, $n \in \mathbb{N}$, is eventually constant.

Exercise 3.3.5. Prove that the sequence $s_n = \left\lfloor \dfrac{5n-(-1)^n}{n/2+5} \right\rfloor$, $n \in \mathbb{N}$, is eventually constant.

Definition 3.3.6. A sequence $\{s_n\}$ *converges* if there exists $L \in \mathbb{R}$ such that for each $\epsilon > 0$ there exists a real number $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \qquad \Rightarrow \qquad |s_n - L| < \epsilon.$$

The number L is called the *limit* of the sequence $\{s_n\}$. We also say that $\{s_n\}$ *converges to* L and write

$$\lim_{n\to\infty} s_n = L \qquad \text{or} \qquad s_n \to L \ \ (n \to \infty).$$

If a sequence does not converge we say that it *diverges.*

Remark 3.3.7. The definition of convergence is a complicated statement. Formally it can be written as:

$$\exists\, L \in \mathbb{R} \ \text{ s.t. } \ \forall\, \epsilon > 0 \ \ \exists\, N(\epsilon) \in \mathbb{R} \ \text{ s.t. } \ \forall\, n \in \mathbb{N}, \ \ n > N(\epsilon) \ \Rightarrow \ |s_n - L| < \epsilon.$$

Exercise 3.3.8. State the negation of the statement in remark 3.3.7.

3.3.1. My informal discussion of convergence. It is easy to agree that the constant sequences are simplest possible sequences. For example the sequence

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
c_n	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

or formally, $c_n = 1$ for all $n \in \mathbb{N}$, is a very simple sequence. No action here! In this case, clearly, $\lim_{n\to\infty} c_n = 1$.

Now, I define $s_n = \dfrac{n-(-1)^n}{n}$, $n \in \mathbb{N}$, and I ask: Is $\{s_n\}$ a constant sequence? Just looking at the first few terms

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
s_n	2	$\frac{1}{2}$	$\frac{4}{3}$	$\frac{3}{4}$	$\frac{6}{5}$	$\frac{5}{6}$	$\frac{8}{7}$	$\frac{7}{8}$	$\frac{10}{9}$	$\frac{9}{10}$	$\frac{12}{11}$	$\frac{11}{12}$	$\frac{14}{13}$	$\frac{13}{14}$	$\frac{16}{15}$	$\frac{15}{16}$	$\frac{18}{17}$

indicates that this sequence is not constant. The table above also indicates that the sequence $\{s_n\}$ is not eventually constant. But imagine that you have a calculator which is capable of displaying only one decimal place. On this calculator the first terms of this sequence would look like:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
s_n	2.0	0.5	1.3	0.8	1.2	0.8	1.1	0.9	1.1	0.9	1.1	0.9	1.1	0.9	1.1

and the next 15 terms would look like:

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
s_n	0.9	1.1	0.9	1.1	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Basically, after the 20-th term this calculator does not distinguish s_n from 1. That is, this calculator leads us to think that $\{s_n\}$ is eventually constant. Why is this? On this calculator all numbers between $0.95 = 1 - 1/20$ and $1.05 = 1 + 1/20$ are represented as 1, and for our sequence we can prove that

$$n \in \mathbb{N}, \quad n > 20 \qquad \Rightarrow \qquad 1 - \frac{1}{20} < s_n < 1 + \frac{1}{20},$$

or, equivalently,

$$n \in \mathbb{N}, \quad n > 20 \qquad \Rightarrow \qquad \bigl|s_n - 1\bigr| < \frac{1}{20}.$$

In the notation of Definition 3.3.6 this means $N(1/20) = 20$.

One can reasonably object that the above calculator is not very powerful and propose to use a calculator that can display three decimal places. Then the terms of $\{s_n\}$ starting with $n = 21$ are

n	21	22	23	24	25	26	27	28	29	30
s_n	1.048	0.955	1.043	0.958	1.040	0.962	1.037	0.964	1.034	0.967

Now the question is: Can we fool this powerful calculator to think that $\{s_n\}$ is eventually constant? Notice that on this calculator all numbers between $0.9995 = 1 - 1/2000$ and $1.0005 = 1 + 1/2000$ are represent as 1. Therefore, in the notation of Definition 3.3.6, we need $N(1/2000)$ such that

$$n \in \mathbb{N}, \quad n > N(1/2000) \qquad \Rightarrow \qquad 1 - \frac{1}{2000} < s_n < 1 + \frac{1}{2000}.$$

An easy calculation shows that $N(1/2000) = 2000$. That is

$$n \in \mathbb{N}, \quad n > 2000 \qquad \Rightarrow \qquad \bigl|s_n - 1\bigr| < \frac{1}{2000}.$$

This is illustrated by the following table

n	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005
s_n	0.999	1.001	0.999	1.001	1.000	1.000	1.000	1.000	1.000	1.000

Hence, even this more powerful calculator is fooled into thinking that $\{s_n\}$ is eventually constant.

In computer science the precision of a computer is measured by the number called the *machine epsilon* (also called *macheps, machine precision* or *unit round-off*). It is the smallest number that gives a number greater than 1 when added to 1.

Now, Definition 3.3.6 can be paraphrased as: A sequence converges if on each computer it appears to be eventually constant. This is the reason why I think that instead of the phrase "a sequence is convergent" we could use the phrase "a sequence is constantish."

3.4. Finding $N(\epsilon)$ for a convergent sequence

Example 3.4.1. Prove that $\displaystyle\lim_{n\to\infty} \frac{2n-1}{n+3} = 2$.

Solution. We prove the given equality using Definition 3.3.6. To do that for each $\epsilon > 0$ we have to find $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \qquad \Rightarrow \qquad \left|\frac{2n-1}{n+3} - 2\right| < \epsilon. \tag{3.4.1}$$

Let $\epsilon > 0$ be given. We can think of n as an unknown in $\left|\dfrac{2n-1}{n+3} - 2\right| < \epsilon$ and solve this inequality for n. To this end first simplify the left-hand side:

$$\left|\frac{2n-1}{n+3} - 2\right| = \left|\frac{2n-1-2n-6}{n+3}\right| = \frac{|-7|}{|n+3|} = \frac{7}{n+3}. \tag{3.4.2}$$

Now, $\dfrac{7}{n+3} < \epsilon$ is much easier to solve for $n \in \mathbb{N}$:

$$\frac{7}{n+3} < \epsilon \quad \Leftrightarrow \quad \frac{n+3}{7} > \frac{1}{\epsilon} \quad \Leftrightarrow \quad n+3 > \frac{7}{\epsilon} \quad \Leftrightarrow \quad n > \frac{7}{\epsilon} - 3. \tag{3.4.3}$$

Now (3.4.3) indicates that we can choose $N(\epsilon) = \frac{7}{\epsilon} - 3$.

Now we have $N(\epsilon)$, but to complete the formal proof, we have to prove implication (3.4.1). The proof follows. Let $n \in \mathbb{N}$ and $n > \frac{7}{\epsilon} - 3$. Then the equivalences in (3.4.3) imply that $\frac{7}{n+3} < \epsilon$. Since by (3.4.3), $\left|\frac{2n-1}{n+3} - 2\right| = \frac{7}{n+3}$, it follows that $\left|\frac{2n-1}{n+3} - 2\right| < \epsilon$. This completes the proof of implication (3.4.1). ☐

Remark 3.4.2. This remark is essential for the understanding of the process described in the following examples. In the solution of Example 3.4.1 we found (in some sense) the smallest possible $N(\epsilon)$. It is important to notice that implication (3.4.1) holds with any larger value for "$N(\epsilon)$." For example, implication (3.4.1) holds if we set $N(\epsilon) = \frac{7}{\epsilon}$. With this new $N(\epsilon)$ we can prove implication (3.4.1) as follows. Let $n \in \mathbb{N}$ and $n > \frac{7}{\epsilon}$. Then $\frac{7}{n} < \epsilon$. Since clearly $\frac{7}{n+3} < \frac{7}{n}$, the last two inequalities imply that $\frac{7}{n+3} < \epsilon$ and we can continue with the same proof as in the solution of Example 3.4.1.

Example 3.4.3. Prove that $\displaystyle\lim_{n\to\infty} \frac{1}{n^3 - n + 1} = 0.$

Solution. We prove the given equality using Definition 3.3.6. To do that for each $\epsilon > 0$ we have to find $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \qquad \Rightarrow \qquad \left| \frac{1}{n^3 - n + 1} - 0 \right| < \epsilon. \tag{3.4.4}$$

Let $\epsilon > 0$ be given. We can think of n as an unknown in $\left| \frac{1}{n^3 - n + 1} - 0 \right| < \epsilon$ and solve this inequality for n. To this end first simplify the left-hand side:

$$\left| \frac{1}{n^3 - n + 1} - 0 \right| = \left| \frac{1}{n^3 - n + 1} \right| = \frac{|1|}{|n^3 - n + 1|} = \frac{1}{n^3 - n + 1}. \tag{3.4.5}$$

Unfortunately $\frac{1}{n^3 - n + 1} < \epsilon$ is not easy to solve for $n \in \mathbb{N}$. Therefore we use the idea from Remark 3.4.2 and replace the quantity $\frac{1}{n^3 - n + 1}$ with a larger quantity. To make a fraction larger we have to make the denominator smaller. Notice that $n^2 - n = n(n-1) \geq n - 1$ for all $n \in \mathbb{N}$. Therefore for all $n \in \mathbb{N}$ we have

$$n^3 - n + 1 = n^3 - (n-1) \geq n^3 - n(n-1) = n(n^2 - n + 1) \geq n.$$

Consequently,

$$\frac{1}{n^3 - n + 1} \leq \frac{1}{n}. \tag{3.4.6}$$

Now, $\frac{1}{n} < \epsilon$ is truly easy to solve for $n \in \mathbb{N}$:

$$\frac{1}{n} < \epsilon \qquad \Leftrightarrow \qquad n > \frac{1}{\epsilon}. \tag{3.4.7}$$

Hence we set $N(\epsilon) = \frac{1}{\epsilon}$.

Now we have $N(\epsilon)$, but to complete the formal proof, we have to prove implication (3.4.4). The proof follows. Let $n \in \mathbb{N}$ and $n > \frac{1}{\epsilon}$. Then the equivalence in (3.4.7) implies that $\frac{1}{n} < \epsilon$. By (3.4.6), $\frac{1}{n^3-n+1} \leq \frac{1}{n}$. The last two inequalities yield that $\frac{1}{n^3-n+1} < \epsilon$. By (3.4.5) it follows that $\left| \frac{1}{n^3-n+1} - 0 \right| < \epsilon$. This completes the proof of implication (3.4.4). □

Example 3.4.4. Prove that $\displaystyle\lim_{n\to\infty} \frac{n^2 - 1}{n^2 - 2n + 2} = 1.$

Solution. We prove the given equality using Definition 3.3.6. To do that for each $\epsilon > 0$ we have to find $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \qquad \Rightarrow \qquad \left| \frac{n^2 - 1}{n^2 - 2n + 2} - 1 \right| < \epsilon. \tag{3.4.8}$$

Let $\epsilon > 0$ be given. We can think of n as an unknown in $\left|\dfrac{n^2-1}{n^2-2n+2} - 1\right| < \epsilon$ and solve this inequality for n. To this end first simplify the left-hand side:

$$\left|\frac{n^2-1}{n^2-2n+2} - 1\right| = \left|\frac{n^2-1-n^2+2n-2}{n^2-2n+2}\right| = \frac{|2n-3|}{n^2-2n+2}. \tag{3.4.9}$$

Unfortunately $\dfrac{|2n-3|}{n^2-2n+2} < \epsilon$ is not easy to solve for $n \in \mathbb{N}$. Therefore we use the idea from Remark 3.4.2 and replace the quantity $\dfrac{|2n-3|}{n^2-2n+2}$ with a larger quantity. Here is one way to discover a desired inequality. We first notice that for all $n \in \mathbb{N}$ the following two inequalities hold

$$|2n-3| \leq 2n \tag{3.4.10}$$

and

$$n^2-2n+2 = \frac{n^2}{2} + \frac{1}{2}\bigl(n^2-4n+4\bigr) = \frac{n^2}{2} + \frac{1}{2}(n-2)^2 \geq \frac{n^2}{2}. \tag{3.4.11}$$

Consequently

$$\frac{|2n-3|}{n^2-2n+2} \leq \frac{2n}{n^2/2} = \frac{4}{n}. \tag{3.4.12}$$

Now, $\dfrac{4}{n} < \epsilon$ is truly easy to solve for $n \in \mathbb{N}$:

$$\frac{4}{n} < \epsilon \qquad \Leftrightarrow \qquad n > \frac{4}{\epsilon}. \tag{3.4.13}$$

Hence we set $N(\epsilon) = \frac{4}{\epsilon}$.

Finally we have $N(\epsilon)$. But to complete the formal proof we have to prove implication (3.4.8). The proof follows. Let $n \in \mathbb{N}$ and $n > \frac{4}{\epsilon}$. Then the equivalence in (3.4.13) implies $\frac{4}{n} < \epsilon$. By (3.4.12), $\frac{|2n-3|}{n^2-2n+2} \leq \frac{4}{n}$. The last two inequalities yield $\frac{|2n-3|}{n^2-2n+2} < \epsilon$. By (3.4.9) it follows that $\left|\frac{n^2-1}{n^2-2n+2} - 1\right| < \epsilon$. This completes the proof of implication (3.4.8). □

Remark 3.4.5. For most sequences $\bigl\{s_n\bigr\}$ a proof of $\lim\limits_{n\to\infty} s_n = L$ based on Definition 3.3.6 should consist from the following steps.

(1) Use algebra to simplify the expression $|s_n - L|$. It is desirable to eliminate the absolute value.
(2) Discover an inequality of the form

$$|s_n - L| \leq b(n) \quad \text{valid for all} \quad n \in \mathbb{N}. \tag{3.4.14}$$

Here $b(n)$ should be a simple function with the following properties:
   (a) $b(n) > 0$ for all $n \in \mathbb{N}$.
   (b) $\lim\limits_{n\to\infty} b(n) = 0$. (Just check this property "mentally.")
   (c) $b(n) < \epsilon$ is easily solvable for n for every $\epsilon > 0$. The solution should be of the form "$n >$ some expression involving ϵ, call it $N(\epsilon)$."
(3) Use inequality (3.4.14) to prove the implication $n \in \mathbb{N}$, $n > N(\epsilon) \Rightarrow |s_n - L| < \epsilon$.

Exercise 3.4.6. Determine the limits (if they exist) of the sequences (e), (f), (g), (h), (i), and (n) in Example 3.1.4. Prove your claims.

Exercise 3.4.7. Determine whether the sequence $\left\{\dfrac{3n+1}{7n-4}\right\}_{n=1}^{\infty}$ converges and, if it converges, give its limit. Provide a formal proof.

Exercise 3.4.8. Determine the limits (if they exist) of the sequences in Example 3.1.5. Prove your claims.

3.5. Two standard sequences

Exercise 3.5.1. Let $a \in \mathbb{R}$ be such that $-1 < a < 1$.

(1) Prove that for all $n \in \mathbb{N}$ we have

$$|a|^n \leq \frac{1}{n\,(1-|a|)}.$$

(2) Prove that

$$\lim_{n\to\infty} a^n = 0.$$

Exercise 3.5.2. Let a be a positive real number. Prove that

$$\lim_{n\to\infty} a^{1/n} = 1.$$

Solution. Let $a > 0$. If $a = 1$, then $a^{1/n} = 1$ for all $n \in \mathbb{N}$. Therefore $\lim_{n\to\infty} a^{1/n} = 1$.

Assume $a > 1$. Then $a^{1/n} > 1$. We shall prove that

$$a^{1/n} - 1 \leq a\,\frac{1}{n} \quad \big(\forall\, n \in \mathbb{N}\big). \tag{3.5.1}$$

Put $x = a^{1/n} - 1 > 0$. Then, by Bernoulli's inequality we get

$$a = (1+x)^n \geq 1 + nx.$$

Consequently, solving for x we get that $x = a^{1/n} - 1 \leq (a-1)/n$. Since $a - 1 < a$, (3.5.1) follows.

Assume $0 < a < 1$. Then $1/a > 1$. Therefore, by already proved (3.5.1), we have

$$\left(\frac{1}{a}\right)^{1/n} - 1 \leq \frac{1}{a}\,\frac{1}{n} \quad \big(\forall\, n \in \mathbb{N}\big).$$

Since $(1/a)^{1/n} = 1/\big(a^{1/n}\big)$, simplifying the last inequality, together with the inequality $a^{1/n} < 1$, yields

$$1 - a^{1/n} \leq \frac{a^{1/n}}{a}\,\frac{1}{n} \leq \frac{1}{a}\,\frac{1}{n} \quad \big(\forall\, n \in \mathbb{N}\big). \tag{3.5.2}$$

As $a < a + 1/a$ and $1/a < a + 1/a$, the inequalities (3.5.1) and (3.5.2) imply

$$\big|a^{1/n} - 1\big| \leq \left(a + \frac{1}{a}\right)\frac{1}{n} \quad \big(\forall\, n \in \mathbb{N}\big). \tag{3.5.3}$$

Let $\epsilon > 0$ be given. Solving $\big(a + 1/a\big)\frac{1}{n} < \epsilon$ for n, reveals $N(\epsilon)$:

$$N(\epsilon) = \left(a + \frac{1}{a}\right)\frac{1}{\epsilon}$$

Now it is easy to prove the implication (Do it as an exercise!)

$$n \in \mathbb{N}, \quad n > \left(a + \frac{1}{a}\right)\frac{1}{\epsilon} \quad \Rightarrow \quad |a^{1/n} - 1| < \epsilon. \qquad \square$$

3.6. Non-convergent sequences

Exercise 3.6.1. Prove that the sequence (d) in Example 3.1.4 does not converge. Use Remark 3.3.7 and Exercise 3.3.8

Exercise 3.6.2. (Prove or Disprove) If $\{s_n\}$ does not converge to L, then there exist $\epsilon > 0$ and $N(\epsilon)$ such that $|s_n - L| \geq \epsilon$ for all $n \geq N(\epsilon)$.

3.7. Convergence and boundedness

Exercise 3.7.1. Consider the following two statements:
(A) The sequence $\{s_n\}$ is bounded.
(B) The sequence $\{s_n\}$ converges.
Is (A)⇒(B) true or false? Is (B)⇒(A) true or false? Justify your answers.

3.8. Algebra of limits of convergent sequences

Exercise 3.8.1. Let $\{s_n\}$ be a sequence in $\mathbb{R}$ and let $L \in \mathbb{R}$. Set $t_n = s_n - L$ for all $n \in \mathbb{N}$.

Prove that $\{s_n\}$ converges to L if and only if $\{t_n\}$ converges to 0.

Exercise 3.8.2. Let $c \in \mathbb{R}$. If $\lim_{n\to\infty} x_n = X$ and $z_n = c\,x_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} z_n = c\,X$.

Exercise 3.8.3. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathbb{R}$. Assume
(a) $\{x_n\}$ converges to 0,
(b) $\{y_n\}$ is bounded,
(c) $z_n = x_n y_n$ for all $n \in \mathbb{N}$.
Prove that $\{z_n\}$ converges to 0.

Exercise 3.8.4. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathbb{R}$. Assume
(a) $\lim_{n\to\infty} x_n = X$,
(b) $\lim_{n\to\infty} y_n = Y$,
(c) $z_n = x_n + y_n$ for all $n \in \mathbb{N}$.
Prove that $\lim_{n\to\infty} z_n = X + Y$.

Exercise 3.8.5. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathbb{R}$. Assume
(a) $\lim_{n\to\infty} x_n = X$,
(b) $\lim_{n\to\infty} y_n = Y$,
(c) $z_n = x_n\, y_n$ for all $n \in \mathbb{N}$.
Prove that $\lim_{n\to\infty} z_n = X\,Y$.

Exercise 3.8.6. If $\lim_{n\to\infty} x_n = X$ and $X > 0$, then there exists a real number N such that $n \geq N$ implies $x_n \geq X/2$.

Exercise 3.8.7. Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Assume
(a) $x_n \neq 0$ for all $n \in \mathbb{N}$,
(b) $\lim_{n\to\infty} x_n = X$,
(c) $X > 0$,

(d) $w_n = \dfrac{1}{x_n}$ for all $n \in \mathbb{N}$.

Prove that $\lim\limits_{n\to\infty} w_n = \dfrac{1}{X}$.

Exercise 3.8.8. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathbb{R}$. Assume

(a) $x_n \neq 0$ for all $n \in \mathbb{N}$,
(b) $\lim\limits_{n\to\infty} x_n = X$,
(c) $\lim\limits_{n\to\infty} y_n = Y$,
(d) $X \neq 0$,
(e) $z_n = \dfrac{y_n}{x_n}$ for all $n \in \mathbb{N}$.

Prove that $\lim\limits_{n\to\infty} z_n = \dfrac{Y}{X}$. (Hint: Use previous exercises.)

Exercise 3.8.9. Prove that $\lim\limits_{n\to\infty} \dfrac{2n^2+n-5}{n^2+2n+2} =$ (insert correct value) by using the results we have proved (Exercises 3.8.2, 3.8.4, 3.8.5, 3.8.7, 3.8.8) and a small trick. You may use Definition 3.3.6 of convergence directly in this problem only to evaluate limit of the special form $\lim\limits_{n\to\infty} \dfrac{1}{n}$.

Remark 3.8.10. The point of Exercise 3.8.9 is to see that the general properties of limits (Exercises 3.8.2, 3.8.4, 3.8.5, 3.8.7, 3.8.8) can be used to reduce complicated situations to a few simple ones, so that when the few simple ones have been done it is no longer necessary to go back to Definition 3.3.6 of convergence every time.

3.9. Convergent sequences and the order in $\mathbb{R}$

Exercise 3.9.1. Let $\{s_n\}$ be a sequence in $\mathbb{R}$. Assume

(a) $\lim\limits_{n\to\infty} s_n = L$.
(b) There exists a real number N_0 such that $s_n \geq 0$ for all $n \in \mathbb{N}$ such that $n > N_0$.

Prove that $L \geq 0$.

Exercise 3.9.2. Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$. Assume

(a) $\lim\limits_{n\to\infty} a_n = K$.
(b) $\lim\limits_{n\to\infty} b_n = L$.
(c) There exists a real number N_0 such that $a_n \leq b_n$ for all $n \in \mathbb{N}$ such that $n > N_0$.

Prove that $K \leq L$.

Exercise 3.9.3. Is the following refinement of Exercise 3.9.1 true? If $\{s_n\}$ converges to L and if $s_n > 0$ for all $n \in \mathbb{N}$, then $L > 0$.

Exercise 3.9.4. Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Assume

(a) $x_n \geq 0$ for all $n \in \mathbb{N}$,
(b) $\lim\limits_{n\to\infty} x_n = X$,
(c) $w_n = \sqrt{x_n}$ for all $n \in \mathbb{N}$.

Prove that $\lim\limits_{n\to\infty} w_n = \sqrt{X}$.

3.10. Squeeze theorem for convergent sequences

Exercise 3.10.1. There are three sequences in this exercise: $\{a_n\}$, $\{b_n\}$ and $\{s_n\}$. Assume the following

(1) The sequence $\{a_n\}$ converges to L.
(2) The sequence $\{b_n\}$ converges to L.
(3) There exists a real number n_0 such that

$$a_n \le s_n \le b_n \quad \text{for all } \; n \in \mathbb{N}, \; n > n_0.$$

Prove that $\{s_n\}$ converges to L.

Exercise 3.10.2. (1) Let $x \ge 0$ and $n \in \mathbb{N}$. Prove the inequality

$$(1+x)^n \ge 1 + n\,x + \frac{n(n-1)}{2}\,x^2.$$

(2) Prove that for all $n \in \mathbb{N}$ we have $1 \le n^{1/n} \le 1 + \dfrac{2}{\sqrt{n}}$.

Hint: Apply the inequality proved in (1) to $\bigl(1 + 2/\sqrt{n}\bigr)^n$.

(3) Prove that the sequence $\{n^{1/n}\}$ converges and determine its limit.

Exercise 3.10.3. (1) Prove that $(n!)^2 \ge n^n$ for all $n \in \mathbb{N}$. Hint: Write

$$\bigl(n!\bigr)^2 = \bigl(1\cdot n\bigr)\bigl(2\cdot(n-1)\bigr)\cdots\bigl((n-1)\cdot 2\bigr)\bigl(n\cdot 1\bigr) = \prod_{k=1}^{n} k\bigl(n-k+1\bigr).$$

Then prove $k\bigl(n-k+1\bigr) \ge n$ for all $k = 1, \ldots, n$.

(2) Prove that

$$\lim_{n\to\infty} \frac{1}{\bigl(n!\bigr)^{1/n}} = 0.$$

3.11. The monotonic convergence theorem

Definition 3.11.1. A sequence $\{s_n\}$ of real numbers is said to be *non-decreasing* if $s_n \le s_{n+1}$ for all $n \in \mathbb{N}$, *strictly increasing* if $s_n < s_{n+1}$ for all $n \in \mathbb{N}$, *non-increasing* if $s_n \ge s_{n+1}$ for all $n \in \mathbb{N}$, *strictly decreasing* if $s_n > s_{n+1}$ for all $n \in \mathbb{N}$. A sequence with any of these properties is said to be *monotonic*.

Exercise 3.11.2. Again a huge task here. Which of the sequences in Examples 3.1.4, 3.1.5, and 3.1.6 are monotonic? Find few monotonic ones in each example. Provide rigorous proofs.

Exercise 3.11.3. (Prove or Disprove) If $\{x_n\}$ is non-increasing, then $\{x_n\}$ converges.

The following two exercises give powerful tools for establishing convergence of a sequence.

Exercise 3.11.4. If $\{s_n\}$ is non-increasing and bounded below, then $\{s_n\}$ converges.

Exercise 3.11.5. If $\{s_n\}$ is non-decreasing and bounded above, then $\{s_n\}$ converges.

PROOF. Assume that the sequence $\{s_n\}$ is non-decreasing and bounded above. Consider the range of the sequence $\{s_n\}$. That is consider the set

$$A = \{s_n \ : \ n \in \mathbb{N}\}.$$

The set A is nonempty and bounded above. Therefore $\sup A$ exists. Put $L = \sup A$.

We will prove that $s_n \to L$ $(n \to \infty)$. Let $\epsilon > 0$ be arbitrary. Since $L = \sup A$ we have

(1) $L \geq s_n$ for all $n \in \mathbb{N}$.
(2) There exists $a_\epsilon \in A$ such that $L - \epsilon < a_\epsilon$.

Since $a_\epsilon \in A$, there exists $N_\epsilon \in \mathbb{N}$ such that $a_\epsilon = s_{N_\epsilon}$. It remains to prove that

$$n \in \mathbb{N}, \quad n > N_\epsilon \quad \Rightarrow \quad |s_n - L| < \epsilon. \tag{3.11.1}$$

Let $n \in \mathbb{N},\ n > N_\epsilon$ be arbitrary. Since we assume that $\{s_n\}$ is non-decreasing, it follows that $s_n \geq s_{N_\epsilon}$. Since $L - \epsilon < a_\epsilon = s_{N_\epsilon} \leq s_n$, we conclude that $L - s_n < \epsilon$. Since $L \geq s_n$, we have $|s_n - L| = L - s_n < \epsilon$. The implication (3.11.1) is proved. □

Exercise 3.11.6. There is a huge task here. Consider the sequences given in Example 3.1.6. Prove that each of these sequences converges and determine its limit.

3.12. Two important sequences with the same limit

In this section we study the sequences defined in (3.1.3) and (3.1.5).

$$E_n = \left(1 + \frac{1}{n}\right)^n, \quad n \in \mathbb{N},$$

$$S_1 = 2 \ \text{ and } \ \big(\forall\, n \in \mathbb{N}\big) \ \ S_{n+1} = S_n + \frac{1}{(n+1)!}.$$

Exercise 3.12.1. Prove by mathematical induction that $S_n \leq 3 - 1/n$ for all $n \in \mathbb{N}$.

Exercise 3.12.2. Prove that the sequence $\{S_n\}$ converges.

Exercise 3.12.3. Let $n, k \in \mathbb{N}$ and $n \geq k$. Use Bernoulli's inequality to prove that

$$\frac{n!}{(n-k)!\, n^k} \geq 1 - \frac{(k-1)k}{n}$$

HINT: Notice that

$$\frac{n!}{n^k (n-k)!} = 1 \cdot \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \geq \left(1 - \frac{k-1}{n}\right)^k.$$

Exercise 3.12.4. The following inequalities hold: $E_1 = S_1$ and for all integers n greater than 1,

$$S_n - \frac{3}{n} < E_n < S_n. \tag{3.12.1}$$

HINT: Let n be an integer greater than 2. Notice that by the Binomial Theorem

$$E_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{1}{n^k} = 1 + 1 + \sum_{k=2}^{n} \frac{n!}{(n-k)!\, n^k} \frac{1}{k!}.$$

Then use Exercise 3.12.3 to prove $E_n > S_n - S_{n-2}/n$. Then use Exercise 3.12.1.

Exercise 3.12.5. The sequences $\{E_n\}$ and $\{S_n\}$ converge to the same limit.

Exercise 3.12.5 justifies the following definition.

Definition 3.12.6. The number e is the common limit of the sequences $\{E_n\}$ and $\{S_n\}$.

Remark 3.12.7. The sequence $\{E_n\}$ is increasing. To prove this claim let $n \in \mathbb{N}$ be arbitrary. Consider the fraction

$$\begin{aligned} \frac{E_{n+1}}{E_n} &= \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n}} = \frac{n+1}{n}\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}} = \frac{n+1}{n}\left(\frac{\frac{n+2}{n+1}}{\frac{n+1}{n}}\right)^{n+1} \\ &= \frac{n+1}{n}\left(\frac{n(n+2)}{(n+1)^2}\right)^{n+1} = \frac{n+1}{n}\left(1-\frac{1}{(n+1)^2}\right)^{n+1} \end{aligned} \tag{3.12.2}$$

Since $-\frac{1}{(n+1)^2} > -1$ for all $n \in \mathbb{N}$, applying Bernoulli's Inequality with $x = -\frac{1}{(n+1)^2}$ we get

$$\left(1-\frac{1}{(n+1)^2}\right)^{n+1} > 1-(n+1)\frac{1}{(n+1)^2} = 1-\frac{1}{n+1}. \tag{3.12.3}$$

The relations (3.12.2) and (3.12.3) imply

$$\frac{E_{n+1}}{E_n} = \frac{n+1}{n}\left(1-\frac{1}{(n+1)^2}\right)^{n+1} > \frac{n+1}{n}\left(1-\frac{1}{n+1}\right) = 1.$$

Thus

$$\frac{E_{n+1}}{E_n} > 1 \quad \text{for all} \quad n \in \mathbb{N},$$

that is the sequence $\{E_n\}$ is increasing.

3.13. Subsequences

Composing functions is a common way how functions interact with each other. Can we compose two sequences? Let $x : \mathbb{N} \to \mathbb{R}$ and $y : \mathbb{N} \to \mathbb{R}$ be two sequences. Does the composition $x \circ y$ make sense? This composition makes sense only if the range of y is contained in $\mathbb{N}$. In this case $y : \mathbb{N} \to \mathbb{N}$. That is the composition $x \circ y$ makes sense only if y is a sequence in $\mathbb{N}$. It turns out that the most important composition of sequences involve increasing sequences in $\mathbb{N}$. In this section the Greek letters μ and ν will always denote increasing sequences of natural numbers.

Definition 3.13.1. A *subsequence* of a sequence $\{x_n\}$ is a composition of the sequence $\{x_n\}$ and an increasing sequence $\{\mu_k\}$ of natural numbers. This composition will be denoted by $\{x_{\mu_k}\}$ or $\{x(\mu_k)\}$.

Remark 3.13.2. The concept of subsequence consists of two ingredients:

- the sequence $\{x_n\}$ (remember it's really a function: $x : \mathbb{N} \to \mathbb{R}$)
- the increasing sequence $\{\mu_k\}$ of natural numbers (remember this is an increasing function: $\mu : \mathbb{N} \to \mathbb{N}$).

The composition $x \circ \mu$ of these two sequences is a new sequence $y : \mathbb{N} \to \mathbb{R}$. The k-th term y_k of this sequence is $y_k = x_{\mu_k}$. Note the analogy with the usual notation for functions: $y(k) = x(\mu(k))$. Usually we will not introduce the new name for a subsequence: we will write $\{x_{\mu_k}\}_{k=1}^{\infty}$ to denote a subsequence of the sequence $\{x_n\}$. Here $\{\mu_k\}_{k=1}^{\infty}$ is and increasing sequence of natural numbers which selects particular elements of the sequence $\{x_n\}$ to be included in the subsequence.

Remark 3.13.3. Roughly speaking, a subsequence of $\{x_n\}$ is a sequence formed by selecting some of the terms in $\{x_n\}$, keeping them in the same order as in the original sequence. It is the sequence $\{\mu_k\}$ of positive integers that does the selecting.

Example 3.13.4. Few examples of increasing sequences in $\mathbb{N}$ are:

(1) $\mu_k = 2k, \quad k \in \mathbb{N}$. (The sequence of even positive integers.)
(2) $\nu_k = 2k - 1, \quad k \in \mathbb{N}$. (The sequence of odd positive integers.)
(3) $\mu_k = k^2, \quad k \in \mathbb{N}$. (The sequence of perfect squares.)
(4) Let j be a fixed positive integer. Set $\nu_k = j + k$ for all $k \in \mathbb{N}$.
(5) The sequence 2, 3, 5, 7, 11, 13, 17, ... of prime numbers. For this sequence no formula for $\{\mu_k\}$ is known.

Exercise 3.13.5. Let $\{\mu_n\}$ be an increasing sequence in $\mathbb{N}$. Prove that $\mu_n \geq n$ for all $n \in \mathbb{N}$.

Exercise 3.13.6. Each subsequence of a convergent sequence is convergent with the same limit.

Remark 3.13.7. The "contrapositive" of Exercise 3.13.6 is a powerful tool for proving that a given sequence does not converge. As an illustration prove that the sequence $\{(-1)^n\}$ does not converge in two different ways: using the definition of convergence and using the "contrapositive" of Exercise 3.13.6.

Exercise 3.13.8 (The Zipper Theorem)**.** Let $\{x_n\}$ be a sequence in $\mathbb{R}$ and let $\{\mu_k\}$ and $\{\nu_k\}$ be increasing sequences in $\mathbb{N}$. Assume

(a) $\{\mu_k : k \in \mathbb{N}\} \bigcup \{\nu_k : k \in \mathbb{N}\} = \mathbb{N}$.
(b) $\{x_{\mu_k}\}$ converges to L.
(c) $\{x_{\nu_k}\}$ converges to L.

Prove that $\{x_n\}$ converges to L.

Example 3.13.9. The sequence (o) in Example 3.1.4 does not converge, but it does have convergent subsequences, for instance the subsequence $\left\{\dfrac{2k}{2k+1}\right\}_{k=1}^{\infty}$ (Here $\mu_k = 2k,\ k \in \mathbb{N}$) and the subsequence $\left\{\dfrac{1}{(2k-1)2k}\right\}_{k=1}^{\infty}$ (Here $\nu_k = 2k-1,\ k \in \mathbb{N}$).

Remark 3.13.10. The notation for subsequences is a little tricky at first. Note that in x_{μ_k} it is k that is the variable. Thus the successive elements of the subsequence are $x_{\mu_1}, x_{\mu_2}, x_{\mu_3}$, etc. To indicate a different subsequence of the same sequence $\{x_n\}_{n=1}^{\infty}$ it would be necessary to change not the variable name, but the selection sequence. For example $\{x_{\mu_k}\}_{k=1}^{\infty}$ and $\{x_{\nu_k}\}_{k=1}^{\infty}$ in Example 3.13.9 are distinct subsequences of $\{x_n\}$. (Thus $\{x_{\mu_k}\}_{k=1}^{\infty}$ and $\{x_{\mu_j}\}_{j=1}^{\infty}$ are the same subsequence of $\{x_n\}_{n=1}^{\infty}$ for exactly the same reason that $x \mapsto x^2\,(x \in \mathbb{R})$ and $t \mapsto t^2\,(t \in \mathbb{R})$ are the same function. To make a different function it's the rule you must change, not the variable name.)

Example 3.13.11. Let $\{x_n\}$ be the sequence defined by

$$x_n = \frac{(-1)^n (n+1)^{(-1)^n}}{n}, \qquad n \in \mathbb{N}.$$

The values of $\{x_n\}$ are

$$-\frac{1}{1\cdot 2}, \frac{3}{2}, -\frac{1}{3\cdot 4}, \frac{5}{4}, -\frac{1}{5\cdot 6}, \frac{7}{6}, -\frac{1}{7\cdot 8}, \frac{9}{8}, -\frac{1}{9\cdot 10}, \frac{11}{10}, \ldots.$$

Exercise 3.13.12. Every sequence has a monotonic subsequence.

HINT: Let $\{x_n\}$ be an arbitrary sequence. Consider the set

$$\mathbb{M} = \{n \in \mathbb{N} : \forall\, k > n \text{ we have } x_k \geq x_n\}.$$

The set $\mathbb{M}$ is either finite or infinite. Construct a monotonic subsequence in each case.

Exercise 3.13.13. Every bounded sequence of real numbers has a convergent subsequence.

3.14. The Cauchy criterion

Definition 3.14.1. A sequence $\{s_n\}$ of real numbers is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists a real number N_ϵ such that

$$\forall\, n, m \in \mathbb{N}, \quad n, m > N_\epsilon \quad \Rightarrow \quad |s_n - s_m| < \epsilon.$$

Exercise 3.14.2. Prove that every convergent sequence is a Cauchy sequence.

Exercise 3.14.3. Prove that every Cauchy sequence is bounded.

Exercise 3.14.4. If a Cauchy sequence has a convergent subsequence, then it converges.

Exercise 3.14.5. Prove that each Cauchy sequence has a convergent subsequence.

Exercise 3.14.6. Prove that a sequence converges if and only if it is a Cauchy sequence.

3.15. Sequences and supremum and infimum

Exercise 3.15.1. Let $A \subset \mathbb{R}$, $A \neq \emptyset$ and assume that A is bounded above. Prove that $a = \sup A$ if and only if

(a) a is an upper bound of A, that is, $a \geq x$, for all $x \in A$;
(b) there exists a sequence $\{x_n\}$ such that

$$x_n \in A \quad \text{for each} \quad n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to\infty} x_n = a.$$

Exercise 3.15.2. Let $A \subset \mathbb{R}$, $A \neq \emptyset$ and assume that A is bounded above. Let $a = \sup A$ and assume that $a \notin A$. Prove that there exists a strictly increasing sequence $\{x_n\}$ such that

$$x_n \in A \quad \text{for each} \quad n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to\infty} x_n = a.$$

Exercise 3.15.3. State and prove the characterization of infimum which is analogous to the characterization of $\sup A$ given in Exercise 3.15.1.

Exercise 3.15.4. State and prove an exercise involving infimum of a set which is analogous to Exercise 3.15.2.

CHAPTER 4

Continuous functions

In this chapter I will always denote a non-empty subset of $\mathbb{R}$.

4.1. The ϵ-δ definition of a continuous function

Definition 4.1.1. A function $f : I \to \mathbb{R}$ is *continuous at a point* $x_0 \in I$ if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$x \in \big(x_0 - \delta(\epsilon), x_0 + \delta(\epsilon)\big) \cap I \quad \Rightarrow \quad |f(x) - f(x_0)| < \epsilon. \tag{4.1.1}$$

The function f is *continuous on* I if it is continuous at each point of I.

Note that the implication in (4.1.1) can be restated as

$$x \in I \;\text{ and }\; |x - x_0| < \delta(\epsilon) \quad \Rightarrow \quad |f(x) - f(x_0)| < \epsilon.$$

Next we restate Definition 4.1.1 using the terminology introduced in Section 2.14. For a function $f : I \to \mathbb{R}$ and a subset $A \subseteq I$ we will use the notation $f(A)$ to denote the set $\big\{y \in \mathbb{R} : \exists\, x \in A \text{ s.t. } f(x) = y\big\} = \big\{f(x) : x \in A\big\}$.

A function $f : I \to \mathbb{R}$ is continuous at a point $x_0 \in I$ if for each neighborhood V of $f(x_0)$ there exists a neighborhood U of x_0 such that

$$f\big(I \cap U\big) \subseteq V.$$

4.2. Finding $\delta(\epsilon)$ for a given function at a given point

In this and the next section we will prove that some familiar functions are continuous.

A general strategy for proving that a given function f is continuous at a given point x_0 is as follows:

Step 1. Simplify the expression $|f(x) - f(x_0)|$ and try to establish a simple connection with the expression $|x - x_0|$. The simplest connection is to discover positive constants δ_0 and K such that

$$x \in I \;\text{ and }\; x_0 - \delta_0 < x < x_0 + \delta_0 \quad \Rightarrow \quad |f(x) - f(x_0)| \le K\,|x - x_0|. \tag{4.2.1}$$

Formulate your discovery as a lemma.

Step 2. Let $\epsilon > 0$ be given. Use the result in Step 1 to define your $\delta(\epsilon)$. For example, if (4.2.1) holds, then $\delta(\epsilon) = \min\big\{\epsilon/K, \delta_0\big\}$.

Step 3. Use the definition of $\delta(\epsilon)$ from Step 2 and the lemma from Step 1 to prove implication (4.1.1).

Example 4.2.1. We will show that the function $f(x) = x^2$ is continuous at $x_0 = 3$. Here $I = \mathbb{R}$ and we do not need to worry about the domain of f.

Step 1. First simplify

$$\big|f(x) - f(x_0)\big| = \big|x^2 - 3^2\big| = \big|(x+3)(x-3)\big| = \big|x+3\big|\,\big|x-3\big|. \tag{4.2.2}$$

Now we notice that if $2 < x < 4$ we have $\bigl|x+3\bigr| = x+3 \le 7$. Thus (4.2.1) holds with $\delta_0 = 1$ and $K = 7$. We formulate this result as a lemma.

Lemma. *Let* $f(x) = x^2$ *and* $x_0 = 3$. *Then*

$$|x-3| < 1 \quad \Rightarrow \quad \bigl|x^2 - 3^2\bigr| < 7|x-3|. \tag{4.2.3}$$

PROOF. Let $|x-3| < 1$. Then $2 < x < 4$. Therefore $x+3 > 0$ and $|x+3| = x+3 < 7$. By (4.2.2) we now have $\bigl|x^2 - 3^2\bigr| < 7|x-3|$. □

Step 2. Now we define $\delta(\epsilon) = \min\bigl\{\epsilon/7, 1\bigr\}$.
Step 3. It remains to prove (4.1.1). To this end, assume $|x-3| < \min\bigl\{\epsilon/7, 1\bigr\}$. Then $|x-3| < 1$. Therefore, by Lemma we have $\bigl|x^2 - 3^2\bigr| < 7|x-3|$. Since by the assumption $|x-3| < \epsilon/7$, we have $7|x-3| < \epsilon$. Now the inequalities

$$\bigl|x^2 - 3^2\bigr| < 7|x-3| \quad \text{and} \quad 7|x-3| < \epsilon$$

imply that $\bigl|x^2 - 3^2\bigr| < \epsilon$. This proves (4.1.1) and completes the proof that the function $f(x) = x^2$ is continuous at $x_0 = 3$.

Exercise 4.2.2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 5x - 8$. Prove that f is continuous at $x_0 = -3$.

Exercise 4.2.3. Prove that the reciprocal function $x \mapsto \dfrac{1}{x}$, $x \neq 0$, is continuous at $x_0 = 1/2$.

Exercise 4.2.4. Let

$$f(x) = x \left\lfloor \frac{1}{x} \right\rfloor \quad \text{for} \quad x \neq 0 \quad \text{and} \quad f(0) = 1.$$

Prove that the function f is continuous at $x_0 = 0$.

Exercise 4.2.5. State carefully what it means for a function f *not* to be continuous at a point x_0 in its domain. (Express this as a formal mathematical statement.)

Exercise 4.2.6. Consider the function f defined in Exercise 4.2.4. Find a point x_0 at which the function f is not continuous. Provide a formal proof. Provide a detailed sketch of the graph of f near the point x_0.

Exercise 4.2.7. Show that the function of Exercise 4.2.2 is continuous on $\mathbb{R}$.

Exercise 4.2.8. Prove that the function $q(x) = 3x^2 + 5$ is continuous at $x = 2$.

Exercise 4.2.9. Prove that $q(x) = 3x^2 + 5$ is continuous on $\mathbb{R}$.

4.3. Familiar continuous functions

Exercise 4.3.1. Let $m, k \in \mathbb{R}$ and $m \neq 0$. Prove that the linear function $\ell(x) = m\,x + k$ is continuous on $\mathbb{R}$.

Exercise 4.3.2. Let $a, b, c \in \mathbb{R}$ and $a \neq 0$. Prove that the quadratic function $q(x) = a\,x^2 + b\,x + c$ is continuous on $\mathbb{R}$.

Exercise 4.3.3. Let $n \in \mathbb{N}$ and let $x, x_0 \in \mathbb{R}$ be such that $x_0 - 1 \le x \le x_0 + 1$. Prove the following inequality

$$\bigl|x^n - x_0^n\bigr| \le n\bigl(|x_0| + 1\bigr)^{n-1}\bigl|x - x_0\bigr|.$$

HINT: First notice that the assumption $x_0 - 1 \le x \le x_0 + 1$ implies that $|x| < |x_0| + 1$. Then use the Mathematical Induction and the identity

$$\left|x^{n+1} - x_0^{n+1}\right| = \left|x^{n+1} - x\,x_0^n + x\,x_0^n - x_0^{n+1}\right|.$$

Exercise 4.3.4. Let $n \in \mathbb{N}$. Prove that the power function $x \mapsto x^n$, $x \in \mathbb{R}$, is continuous on $\mathbb{R}$.

Exercise 4.3.5. Let $n \in \mathbb{N}$ and let $a_0, a_1, \ldots, a_n \in \mathbb{R}$ with $a_n \neq 0$. Prove that the n-th order polynomial

$$p(x) = a_0 + a_1\,x + \cdots + a_{n-1}\,x^{n-1} + a_n\,x^n$$

is a continuous function on $\mathbb{R}$.

Exercise 4.3.6. Prove that the reciprocal function $x \mapsto \dfrac{1}{x}$, $x \neq 0$, is continuous on its domain.

Exercise 4.3.7. Prove that the square root function $x \mapsto \sqrt{x}$, $x \ge 0$, is continuous on its domain.

Exercise 4.3.8. Let $n \in \mathbb{N}$ and let x and a be positive real numbers. Prove that

$$\left|\sqrt[n]{x} - \sqrt[n]{a}\right| \le \frac{\sqrt[n]{a}}{a}\,\left|x - a\right|.$$

HINT: Notice that the given inequality is equivalent to

$$b^{n-1}\,\left|y - b\right| \le \left|y^n - b^n\right|, \quad y, b > 0.$$

This inequality can be proved using Exercise 2.7.7 (with $a = 1$ and $x = y/b$).

Exercise 4.3.9. Let $n \in \mathbb{N}$. Prove that the n-th root function $x \mapsto \sqrt[n]{x}$, $x \ge 0$, is continuous on its domain.

4.4. Various properties of continuous functions

Exercise 4.4.1. Let $f : I \to \mathbb{R}$ be continuous at $x_0 \in I$ and let y be a real number such that $f(x_0) < y$. Then there exists $\alpha > 0$ such that

$$x \in I \cap (x_0 - \alpha, x_0 + \alpha) \quad \Rightarrow \quad f(x) < y.$$

Illustrate with a diagram.

Exercise 4.4.2. Let $f : I \to \mathbb{R}$ be a continuous function on I. Let S be a non-empty bounded above subset of I such that $u = \sup S$ belongs to I. Let $y \in \mathbb{R}$. Prove: If $f(x) \le y$ for each $x \in S$, then $f(u) \le y$.

The following exercise establishes a connection between continuous functions and convergent sequences.

Exercise 4.4.3. Let $f : I \to \mathbb{R}$ be continuous at $x_0 \in I$. Let $\{t_n\}$ be a sequence in I that converges to $x_0 \in I$. Then $f\,(t_n) \to f(x_0)$ as $n \to \infty$.

Exercise 4.4.4. Let $f : I \to \mathbb{R}$ be continuous at $x_0 \in I$. Let $\{t_n\}$ be a sequence in I that converges to $x_0 \in I$. Assume that there is a real number y such that $f(t_n) \le y$ for all $n \in \mathbb{N}$. Then $f(x_0) \le y$.

Exercise 4.4.5. Let $f : I \to \mathbb{R}$ be continuous at $x_0 \in I$. Let $\{x_n\}$ be a sequence in I that converges to $x_0 \in I$. Assume that there is a real number y such that $f(t_n) \ge y$ for all $n \in \mathbb{N}$. Then $f(x_0) \ge y$.

4.5. Algebra of continuous functions

All exercises in this section have the same structure. With the exception of Exercise 4.5.3, there are three functions in each exercise: f, g and h. The function h is always related in a simple (green) way to the functions f and g. Based on the given (green) information about f and g you are asked to prove a claim (red) about the function h.

Exercise 4.5.1. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be given functions with a common domain. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = f(x) + g(x), \qquad x \in I.$$

(a) If f and g are continuous at $x_0 \in I$, then h is continuous at x_0.
(b) If f and g are continuous on I, then h is continuous on I.

Exercise 4.5.2. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be given functions with a common domain. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = f(x)g(x), \qquad x \in I.$$

(a) If f and g are continuous at $x_0 \in I$, then h is continuous at x_0.
(b) If f and g are continuous on I, then h is continuous on I.

Exercise 4.5.3. Let $g : I \to \mathbb{R}$ be a given functions such that $g(x) \neq 0$ for all $x \in I$. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = \frac{1}{g(x)}, \qquad x \in I.$$

(a) If g is continuous at $x_0 \in I$, then h is continuous at x_0.
(b) If g is continuous on I, then h is continuous on I.

Exercise 4.5.4. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be given functions with a common domain. Assume that $g(x) \neq 0$ for all $x \in I$. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = \frac{f(x)}{g(x)}, \qquad x \in I.$$

(a) If f and g are continuous at $x_0 \in I$, then h is continuous at x_0.
(b) If f and g are continuous on I, then h is continuous on I.

Exercise 4.5.5. Let I and J be non-empty subsets of $\mathbb{R}$. Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be given functions. Assume that the range of f is contained in J. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = g(f(x)), \qquad x \in I.$$

(a) If f is continuous at $x_0 \in I$ and g is continuous at $f(x_0) \in J$, then h is continuous at x_0.
(b) If f is continuous on I and g is continuous on J, then h is continuous on I.

4.6. Continuous functions on a closed bounded interval $[a, b]$

In this section we assume that $a, b \in \mathbb{R}$ and $a < b$.

Exercise 4.6.1. Let $\alpha, \beta, \gamma \in \mathbb{R}$. If $\alpha\beta \leq 0$, then $\alpha\gamma \leq 0$ or $\beta\gamma \leq 0$.

Exercise 4.6.2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a)f(b) \le 0$, then there exists $z \in [a, b]$ such that $f(z) = 0$.

HINT 1: Use Cantor's intersection theorem. Define a sequence of closed intervals $[a_n, b_n]$, $n \in \mathbb{N}$, such that

$$[a_n, b_n] \subseteq [a, b], \qquad [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n], \qquad b_n - a_n = (b - a)/2^{n-1},$$

and, most importantly, $f\big(a_n\big)f\big(b_n\big) \le 0$ for all $n \in \mathbb{N}$.

HINT 2: Assume that $f(a) < 0$ and $f(b) > 0$ and consider the set

$$W = \big\{ w \in [a, b) \, : \, f(x) < 0 \; \forall x \in [a, w] \big\}.$$

Exercise 4.6.3. Let $f : D \to \mathbb{R}$ be a function defined on a nonempty set D. If $D = A \cup B$, then one of the following two statements hold:

(a) For each $x \in D$ there exists $y \in A$ such that $f(x) \le f(y)$.
(b) For each $x \in D$ there exists $y \in B$ such that $f(x) \le f(y)$.

Exercise 4.6.4. Let $f : [a, b] \to \mathbb{R}$ be a function defined on $[a, b]$. Then for each $\eta > 0$ there exists $c, d \in [a, b]$ such that $0 < d - c < \eta$ and for each $x \in [a, b]$ there exists $y \in [c, d]$ such that $f(x) \le f(y)$.

HINT: Use a part of the hint for Exercise 4.6.2 and Exercise 4.6.3.

Exercise 4.6.5. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $w \in [a, b]$ such that $f(x) \le f(w)$ for all $x \in [a, b]$.

HINT 1: Use Cantor's intersection theorem, a part of the hint for Exercise 4.6.2 and Exercise 4.6.3.

HINT 2: Consider the set

$$W = \Big\{ w \in [a, b) \, : \, \exists \, z \in (w, b] \; \text{ such that } \; f(x) < f(z) \;\; \forall \, x \in [a, w] \; \Big\}.$$

Here $[a, a]$ denotes the set $\{a\}$. Prove that the set W has the following property: If $[a, v) \subseteq W$, with $a < v$, and if there exists $t \in [a, b]$ such that $f(t) > f(v)$, then $v \in W$.

Exercise 4.6.6. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $v \in [a, b]$ such that $f(v) \le f(x)$ for all $x \in [a, b]$.

HINT: Use Exercise 4.6.5.

Exercise 4.6.7. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the range of f is a closed bounded interval.

HINT: Use Exercises 4.6.5, 4.6.6, and 4.6.2.

Exercise 4.6.8. Consider the function $f(x) = x^5 - x$, $x \in \mathbb{R}$.

(a) Prove that 1 is in the range of f.
(b) Prove that the range of f equals $\mathbb{R}$.

Definition 4.6.9. A function f is *increasing* on an interval I if $x, y \in I$ with $x < y$ imply $f(x) < f(y)$. A function f is *decreasing* if $x, y \in I$ with $x < y$ imply $f(x) > f(y)$. A function which is increasing or decreasing is said to be *strictly monotonic*.

Exercise 4.6.10. If f is continuous and increasing on $[a, b]$ or continuous and decreasing on $[a, b]$, then for each y between $f(a)$ and $f(b)$ there is exactly one $x \in [a, b]$ such that $f(x) = y$.

Exercise 4.6.11. Let $f(x) = x^3 + x$, $x \in \mathbb{R}$. Prove that f has an inverse. That is, prove that for each $y \in \mathbb{R}$ there exists unique $x \in \mathbb{R}$ such that $f(x) = y$.