ON A ZERO OF A CONTINUOUS FUNCTION

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In this note a and b are real numbers and $a < b$.

Definition 1. A function $f : [a,b] \to \mathbb{R}$ is *continuous at a point* $x_0 \in [a,b]$ if for each $\epsilon > 0$ there exists $\delta(\epsilon, x_0) > 0$ such that

$$x \in \big(x_0 - \delta(\epsilon, x_0), x_0 + \delta(\epsilon, x_0)\big) \cap [a,b] \quad \Rightarrow \quad |f(x) - f(x_0)| < \epsilon.$$

Theorem. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. If $f(a) > 0$ and $f(b) < 0$, then there exists $c \in [a,b]$ such that $f(c) = 0$.*

Proof. Assume $f(a) > 0$ and $f(b) < 0$.

Step 1. Set

$$W = \big\{ x \in [a,b] : \quad f(x) > 0 \big\}.$$

Clearly $a \in W$, $b \notin W$ and $W \subset [a,b]$. Therefore, $c = \sup W$ exists by the Completeness Axiom. Since $a \in W$ and b is an upper bound for W we have $c \in [a,b]$.

Step 2. Here we show that W does not have a maximum. Let $v \in W$ be arbitrary. Then $v < b$ and $f(v) > 0$. Set $\epsilon_1 = f(v)/2$. Since $\epsilon_1 > 0$ and f is continuous at v there exists $\delta_1 = \delta(\epsilon_1, v) > 0$ such that

$$x \in [a,b] \cap \big(v - \delta_1, v + \delta_1\big) \quad \Rightarrow \quad f(v) - \epsilon_1 < f(x) < f(v) + \epsilon_1. \tag{1}$$

Set $\mu = \frac{1}{2}\min\big\{\delta_1, b - v\big\}$. Then $\mu > 0$ and $v + \mu < b$ and $v + \mu < v + \delta_1$. It follows from (1) that $f(v+\mu) > f(v) - \epsilon_1 = f(v)/2 > 0$. Thus $v + \mu \in W$. Since $v + \mu > v$, we proved that v is not a maximum of W.

Step 3. Since W does not have a maximum, $c \notin W$. Since $c \in [a,b]$ and $c \notin W$ we conclude that $f(c) \leq 0$.

Step 4. Here we show that $f(c) \geq 0$. Let $\epsilon > 0$ be arbitrary. Since f is continuous at c, there exists $\delta(\epsilon, c) > 0$ such that

$$x \in [a,b] \cap \big(c - \delta(\epsilon, c), c + \delta(\epsilon, c)\big) \quad \Rightarrow \quad f(c) - \epsilon < f(x) < f(c) + \epsilon. \tag{2}$$

Since $c = \sup W$ and $\delta(\epsilon, c) > 0$ there exists $w \in W$ such that

$$c - \delta(\epsilon, c) < w < c.$$

Now (2) and $f(w) > 0$ yield $0 < f(w) < f(c) + \epsilon$. Since $\epsilon > 0$ was arbitrary, we proved that $f(c) > -\epsilon$ for all $\epsilon > 0$. Consequently $f(c) \geq 0$.

Step 5. In Step 3 we proved $f(c) \leq 0$. In Step 4 we proved $f(c) \geq 0$. Thus $f(c) = 0$. This completes the proof. $\square$

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