MATH 312 Assignment 1 May 6, 2011

Name ____________________

Problem 1. Prove: For every $n \in \mathbb{N}$ there exist a unique $m \in \mathbb{N}$ and a unique $k \in \{0, 1, \ldots, 2(m-1)\}$ such that $n = m^2 - k$.
Uniqueness has been proved on the blog. Just prove the existence.

Problem 2. Let $n \in \mathbb{N}$ be arbitrary. Let $n = m^2 - k$ where $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, 2(m-1)\}$ are the unique numbers whose existence has been proved in Problem 1. Define

$$S(n) := \bigl(1 + |m - 1 - k|, 1 + k\bigr).$$

Prove that the mapping $S : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a surjection.

Problem 3. Let $a \in \mathbb{R}$ be such that $-1 < a < 1$. Let $n \in \mathbb{N}$. Use Bernoulli's inequality to prove that

$$|a|^n \leq \frac{|a|}{n(1 - |a|) + |a|}.$$

Definition. Given a set S, the *power set* (or *powerset*) of S, denoted by $\mathcal{P}(S)$ or 2^S, is the set of all subsets of S.

Problem 4. Prove that $\mathcal{P}(\mathbb{N})$ is infinite, but not countable. HINT: This problem is closely related to Exercise 2.9.7. You can either use the same method of proof, or establish a bijection between $\mathcal{P}(\mathbb{N})$ and $\{0, 1\}^{\mathbb{N}}$.

Problem 5. Prove that there is a bijection between $\mathbb{Q} \cap (0, 1)$ and $\mathbb{Q}$. COMMENT: The "easiest" way to prove this is to find a formula for which you can prove that it is a bijection.

Problem 6. Let $n \in \mathbb{N}$ be arbitrary. Let $x_1, \ldots, x_n$ be arbitrary nonnegative real numbers. Prove

$$\left(\frac{x_1 + \cdots + x_n}{n}\right)^n \geq x_1 \cdots x_n.$$

This is the inequality of arithmetic and geometric means since it can be rewritten as

$$\underbrace{\frac{x_1 + \cdots + x_n}{n}}_{\text{arithmetic mean}} \geq \underbrace{\sqrt[n]{x_1 \cdots x_n}}_{\text{geometric mean}} .$$

Remark. In each of the problems you are supposed to present your own work only. Problem 6 is an exception to this general rule. You are allowed to look for proofs on the Internet. A proof by Mathematical Induction is expected. Your proof has to be complete. You are allowed to use only what has been already proved in class. You have to present your proof in the format that has been established in class.