ON THE MAXIMUM OF A CONTINUOUS FUNCTION

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In this note a and b are real numbers such that $a < b$ and $f : [a,b] \to \mathbb{R}$ is a function defined on $[a,b]$.

Definition 1. Let $f : [a,b] \to \mathbb{R}$ be a given function. If $z \in [a,b]$ and $f(z) \geq f(x)$ for all $x \in [a,b]$, then the value $f(z)$ is called a *maximum of* f.

Definition 2. A function $f : [a,b] \to \mathbb{R}$ is *continuous at a point* $x_0 \in [a,b]$ if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon, x_0) > 0$ such that

$$x \in [a,b] \ \text{ and } \ |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \epsilon.$$

The function f is *continuous on* $[a,b]$ if it is continuous at each point $x_0 \in [a,b]$.

Let $\alpha, \beta \in [a,b], \alpha \leq \beta$. In this note we say that the function f is *dominated on* $[\alpha, \beta]$ if there exists $z_0 \in [a,b]$ such that $f(x) < f(z_0)$ for all $x \in [\alpha, \beta]$; see Fig. 1 and 2.

The following two lemmas give two simple properties of domination.

Lemma 1. *Let $\alpha, \alpha_1, \beta, \beta_1, \in [a,b]$ and $\alpha \leq \alpha_1 \leq \beta_1 \leq \beta$. If f is dominated on $[\alpha, \beta]$, then f is dominated on $[\alpha_1, \beta_1]$.*

Proof. If f is dominated on $[\alpha, \beta]$, then for some $z_0 \in [a,b]$ we have $f(x) < f(z_0)$ for all $x \in [\alpha, \beta]$. Since $[\alpha_1, \beta_1] \subseteq [\alpha, \beta]$ we have $f(x) < f(z_0)$ for all $x \in [\alpha_1, \beta_1]$. Hence f is dominated on $[\alpha_1, \beta_1]$. □

Lemma 2. *Let $\alpha, \beta, \gamma \in [a,b]$ and $\alpha \leq \beta \leq \gamma$. If f is dominated on both intervals $[\alpha, \beta]$ and $[\beta, \gamma]$, then f is dominated on the interval $[\alpha, \gamma]$.*

Proof. Assume that f is dominated on both intervals $[\alpha, \beta]$ and $[\beta, \gamma]$. Then there exists $z_0, z_1 \in [a,b]$ such that $f(x) < f(z_0)$ for all $x \in [\alpha, \beta]$ and $f(x) < f(z_1)$ for all $x \in [\beta, \gamma]$. Set

$$z_2 := \begin{cases} z_0 & \text{if} \quad f(z_1) \leq f(z_0), \\ z_1 & \text{if} \quad f(z_0) < f(z_1). \end{cases}$$

Then $f(z_1) \leq f(z_2)$ and $f(z_0) \leq f(z_2)$. Therefore, $f(x) < f(z_2)$ for all $x \in [\alpha, \gamma]$. Hence f is dominated on $[\alpha, \gamma]$. □

In the following three lemmas we prove properties of domination which require continuity of the function f at a point.

Lemma 3. *Let $d \in [a,b]$. If f is continuous at d and $f(d)$ is not a maximum of f, then there exists $\eta > 0$ such that f is dominated on the interval $[d - \eta, d + \eta] \cap [a,b]$.*

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Proof. This proof is illustrated in Fig. 3. Suppose that f is continuous at d and $f(d)$ is not a maximum of f. Then there exists $y \in [a,b]$ such that $f(d) < f(y)$. Set

$$\epsilon_0 = \frac{f(y) - f(d)}{2} > 0.$$

Since f is continuous at d, there exists $\delta_0 = \delta(\epsilon_0, d) > 0$ such that

$$x \in [a,b] \ \text{ and } \ |x-d| < \delta_0 \quad \Rightarrow \quad f(d) - \epsilon_0 < f(x) < f(d) + \epsilon_0. \tag{1}$$

Choose $\eta > 0$ such that $\eta < \delta_0$. Then clearly

$$x \in [d-\eta, d+\eta] \cap [a,b] \quad \Rightarrow \quad x \in [a,b] \ \text{ and } \ |x-d| < \delta_0,$$

and therefore, by (1),

$$x \in [d-\eta, d+\eta] \cap [a,b] \quad \Rightarrow \quad f(d) - \epsilon_0 < f(x) < f(d) + \epsilon_0.$$

Since

$$f(d) + \epsilon_0 = f(d) + \frac{f(y)-f(d)}{2} = \frac{f(y)+f(d)}{2} < \frac{f(y)+f(y)}{2} = f(y),$$

we have

$$x \in [d-\eta, d+\eta] \cap [a,b] \quad \Rightarrow \quad f(x) < f(y).$$

That is,

$$f(x) < f(y) \quad \text{for all} \quad x \in [d-\eta, d+\eta] \cap [a,b].$$

This proves that f is dominated on the interval $[d-\eta, d+\eta] \cap [a,b]$. □

Lemma 4. *Let $\alpha, \beta \in [a,b)$ and $\alpha \leq \beta$. If f is continuous at β and if f is dominated on $[\alpha,\beta]$, then there exists $\mu > 0$ such that $\beta + \mu < b$ and f is dominated on $[\alpha, \beta+\mu]$.*

Proof. Assume that f is dominated on $[\alpha,\beta]$. Let $z_0 \in [a,b]$ be such that $f(x) < f(z_0)$ for all $x \in [\alpha,\beta]$. In particular, $f(\beta) < f(z_0)$. Thus $f(\beta)$ is not a maximum of f. By Lemma 3 there exists $\eta > 0$ such that f is dominated on $[\beta-\eta, \beta+\eta] \cap [a,b]$. Since $\beta < b$ we set $\mu = \min\{\eta, (b-\beta)/2\} > 0$. Then $\beta + \mu < b$ and thus $[\beta, \beta+\mu] \subseteq [\beta-\eta, \beta+\eta] \cap [a,b]$. As f is dominated on $[\beta-\eta, \beta+\eta] \cap [a,b]$ Lemma 1 implies that f is also dominated on $[\beta, \beta+\mu]$. Since by assumption f is dominated on $[\alpha,\beta]$, Lemma 2 implies that f is dominated on $[\alpha, \beta+\mu]$. □

Lemma 5. *Let $d \in (a,b]$. Assume*

(i) *f is dominated on $[a,\beta]$ for every $\beta < d$;*
(ii) *f is continuous at d;*
(iii) *$f(d)$ is not a maximum of f.*

Then f is dominated on $[a,d]$.

Proof. Assume (i),(ii) and (iii). By Lemma 3 there exists $\eta > 0$ such that f is dominated on the interval $[d-\eta, d+\eta] \cap [a,b]$. Since $a < d$, the number $\nu = \min\{\eta, (d-a)/2\}$ is positive. By the definition of ν we have $a < d - \nu$ and thus $[d-\nu, d] \subseteq [d-\eta, d+\eta] \cap [a,b]$. Since f is dominated on $[d-\eta, d+\eta] \cap [a,b]$, by Lemma 1 f is also dominated on $[d-\nu, d]$. Since $a < d - \nu < d$, the assumption

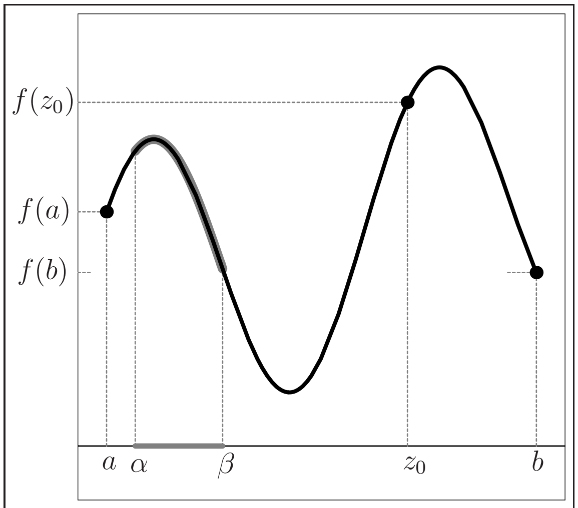

FIG. 1. A dominated interval

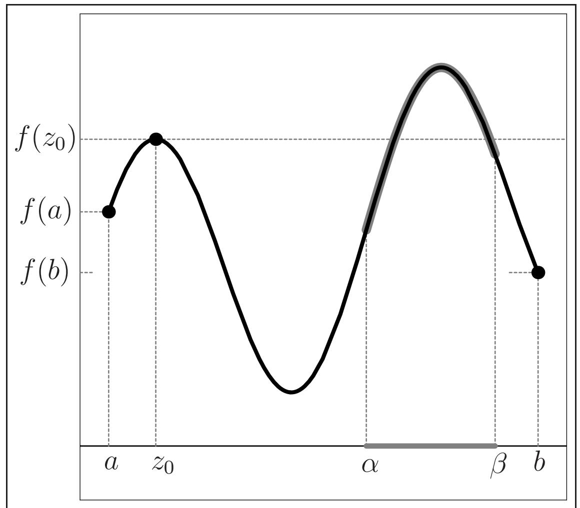

FIG. 2. Not a dominated interval

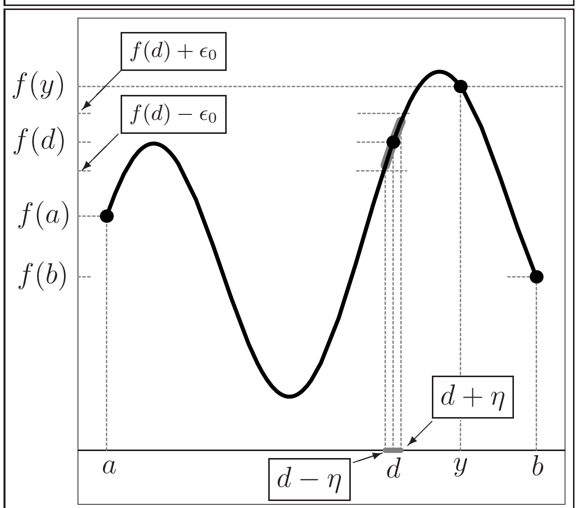

FIG. 3. $f(d)$ is not a maximum

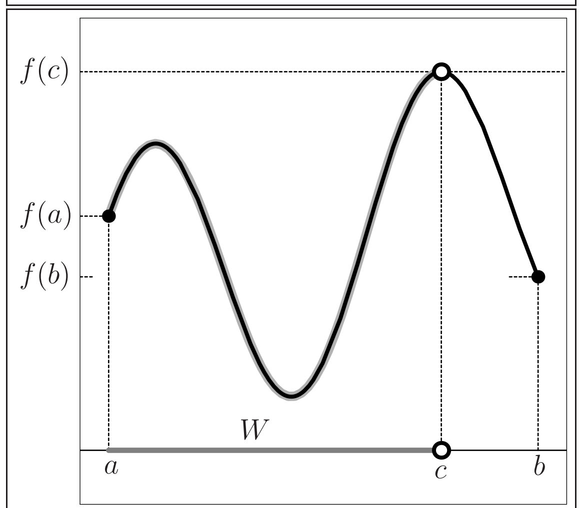

FIG. 4. $c = \sup W$, $f(c)$ is a maximum

(i) implies that f is dominated on $[a, d - \nu]$. Since f is dominated on both intervals $[a, d-\nu]$ and $[d-\nu, d]$, Lemma 2 implies that f is dominated on $[a, d]$. $\square$

The next corollary is a partial contrapositive of the preceding lemma.

Corollary 6. *Let $d \in (a, b]$. Assume*

(i) *f is dominated on $[a, \beta]$ for every $\beta < d$;*
(ii) *f is continuous at d;*
(iii) *f is not dominated on $[a, d]$.*

Then $f(d)$ is a maximum of f.

Theorem. *Let $a, b \in \mathbb{R}$, $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Then there exists $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$.*

Proof. Case I. The value $f(a)$ is a maximum of f. In this case we can set $c = a$.

Case II. The value $f(a)$ is not a maximum of f. Define (see Fig. 4)

$$W = \Big\{ \beta \in [a,b] : \quad f \ \text{ is dominated on } \ [a,\beta] \ \Big\}.$$

Notice that f is not dominated on $[a,b]$ since the statement

$$\exists\, z_0 \in [a,b] \quad \text{such that} \quad \forall\, x \in [a,b] \quad f(x) < f(z_0)$$

is false. Therefore $b \notin W$.

Step 1. Since $f(a)$ is not a maximum of f, by Lemma 3 there exists $\eta_1 > 0$ such that f is dominated on $[a, a+\eta_1] \cap [a,b]$. As $[a,b]$ is not dominated, $a + \eta_1 < b$. Thus f is dominated on $[a, a+\eta_1] \cap [a,b] = [a, a+\eta_1]$. Hence $a + \eta_1 \in W$. Consequently, $W \neq \emptyset$. Since $W \subseteq [a,b)$, W is bounded. Hence $c = \sup W$ exists by the Completeness Axiom. Since b is an upper bound of W and $a + \eta_1 \in W$, we have $a < c \leq b$.

Step 2. Let $\beta \in W$. Then $\beta \in [a,b)$ and f is dominated on $[a,\beta]$. Since f is continuous at β, Lemma 4 implies that there exists $\eta > 0$ such that f is also dominated on $[a, \beta + \eta]$. Hence $\beta + \eta \in W$. This proves that W does not have a maximum. Therefore $c \notin W$.

Step 3. Here we show that $[a,c) \subseteq W$. Let $\beta \in [a,c)$ be arbitrary. Since $\beta < c$ and $c = \sup W$, β is not an upper bound of W. Hence, there exists $\gamma \in W$ such that $\beta < \gamma < c$. Since f is dominated on $[a,\gamma]$ and $[a,\beta] \subseteq [a,\gamma]$, Lemma 1 implies that f is dominated on $[a,\beta]$. Hence $\beta \in W$. This proves $[a,c) \subseteq W$.

Step 4. By Step 2, $c \notin W$. Therefore f is not dominated on $[a,c]$. By Step 3 we have $[a,c) \subseteq W$. Therefore f is dominated on $[a,\beta]$ for every $\beta \in [a,c)$. Now Corollary 6 implies that $f(c)$ is a maximum of f.

The proof is complete. □