MATH 312 Assignment 2 August 11, 2011

Name ______________________

Problem 1. Let $\{I_k, k \in \mathbb{N}\}$ be a family of open bounded intervals in $\mathbb{R}$. Prove the following implication: If the intersection of the intervals $I_k, k \in \mathbb{N}$, is nonempty, then the union of these intervals is also an open interval. (HINT: this is a sup − inf problem.)

Problem 2. Let A and B be nonempty subsets of $\mathbb{R}$. Define the set $A+B$ to be the set of all real numbers x for which there exist $a \in A$ and $b \in B$ such that $x = a + b$, i.e.,

$$A + B = \bigl\{x \in \mathbb{R} : \exists a \in A \text{ and } \exists b \in B \text{ such that } x = a + b\bigr\}.$$

(a) Work out the set $A + B$ in each of the following cases:

(i) $A = (0, 1],\ B = [-1, 0)$; (ii) $A = [0, 1],\ B = \{1, 2, 3\}$; (iii) $A = (0, 1),\ B = \{1, 2, 3\}$;

(b) Prove that A and B are bounded above if and only if $A + B$ is bounded above.

(c) If $A + B$ is bounded above, then $\sup(A + B) = \sup A + \sup B$.

Problem 3. Let $a < b$. Prove that the closed interval $[a, b]$ has the Heine-Borel property: Let $\{I_k, k \in \mathbb{N}\}$ be a family of open intervals in $\mathbb{R}$. The following implication holds: If

$$[a, b] \subset \bigcup_{k \in \mathbb{N}} I_k,$$

then there exists $n \in \mathbb{N}$ such that

$$[a, b] \subset \bigcup_{k=1}^{n} I_k.$$

HINT: Consider the set

$$S = \Bigl\{x \in [a, b] \ : \ \exists\, k \in \mathbb{N} \text{ such that } [a, x] \subset \bigcup_{j=1}^{k} I_j\Bigr\}.$$

Definition 1. A family $\mathcal{A}$ of sets is said to be *pairwise disjoint* or *mutually disjoint* for arbitrary $A, B \in \mathcal{A}$ implies $A = B$ or $A \cap B = \emptyset$.

In the problem below, we call a subset A of $\mathbb{R}$ an open interval if there exist $a, b \in \mathbb{R}$ such that $A = (a, b)$.

Problem 4. Let $\mathcal{I}$ be an infinite family of open mutually disjoint intervals. Prove that $\mathcal{I}$ is countable.

Problem 5. Use ϵ-δ definition of continuity to prove that the function

$$f(x) = \frac{x}{x^2 + 1}, \qquad x \in \mathbb{R},$$

is continuous on its domain.

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $c \in \mathbb{R}$ and define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(cx), x \in \mathbb{R}$. Prove the following implication: If f is continuous, then g is continuous. Is the converse true? Give a complete answer with proofs.