Proofs in Elementary Analysis

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Contents

CHAPTER 1

Introductory Material

1.1. Goals

- To provide a systematic foundation of some basic concepts encountered in calculus, particularly those associated with the structure of the real numbers and notions of limit and continuity for real-valued functions.
- To introduce students to the nature and role of proofs in mathematics. Specifically we assert that the only way to understand proofs is to construct proofs on your own.
- To develop ability to critically read and judge the correctness and the completeness of mathematical reasoning.
- To develop a skill in the clear and precise presentation of mathematical reasoning.

1.2. Strategies

How to get started towards a solution of a problem?

(1) Illustrate the problem with several examples.
(2) Make sure that you understand the terminology used in the problem. Review all relevant definitions.
(3) Can you restate the problem as an implication? (Clearly identify the assumptions and the conclusion of the implication.)
(4) Identify problems done in class that are in some sense related to the problem that you are working on. Review proofs of those problems.
(5) Try to identify tools that can be used in the solution of the problem.
(6) If you can not solve the given problem, try to formulate a related simpler problem that you can solve. For example, try to solve a special case.
(7) Be flexible. Have in mind that there are many ways to approach each problem.
(8) Keep a detailed written record of your work.

How to avoid mistakes?

(1) Write your solution out carefully. Include justifications for all arguments that you use.
(2) Read your solution critically after a day or two. Is everything that you use in your proof justified.
(3) Imagine that a skeptic is reading your proof. Can you answer all sceptic's question?

1.3. Mathematics and logic

Proofs in mathematics are logical arguments. The purpose of this section is to remind you briefly of some of the common strategies of proof, and of the facts of logical equivalence of certain kinds of statements on which these strategies depend.

1.3.1. Implications. Most theorems in mathematics can be stated as *implications* (or *conditional statements*). An implication is a statement of the form "If P, then Q." Here P an Q are simple statements that can be either true or false. The statement "If P, then Q." is symbolically written as $P \Rightarrow Q$.

The implication $P \Rightarrow Q$ is false when P is true and Q is false, and true otherwise. This is summarized in the truth table on the right.

P	Q	$P \Rightarrow Q$
F	F	T
F	T	T
T	F	F
T	T	T

In the implication $P \Rightarrow Q$, P is called the *hypothesis* (or *premise*) and Q is called the *conclusion* (or *consequence*).

To make mathematical language more colorful we use a great variety of different ways of saying: "If P, then Q." Here are some of the most common:

Q when P.	Q follows from P.	P is sufficient for Q.
Q if P.	Q whenever P.	Q is necessary for P.
Q by P.	P only if Q.	A sufficient condition for Q is P.
When P, Q.	P implies Q.	A necessary condition for P is Q.
If P, Q.	By P, Q.	Q provided that P.

Try constructing different ways of saying "If P, then Q." using some everyday statements P and Q, and some mathematical statements P and Q suggested below.

P	Q
It rains.	WWU's Red Square is wet.
You get 100% on the final.	You will get an A.
It is sunny today.	We will go to the beach.
I get to the camp first.	I will raise the flag.
n is a positive integer.	$2\,n^2$ is not a square number.
An integer n is divisible by 9.	The sum of the digits in n is divisible by 9.
n is a positive integer.	$n(n+1)$ is even.
$x^2 < x$	$x > 0$ and $x < 1$.

Starting with an implication "If P, then Q." it is possible to produce three more implications by shuffling the order and possibly introducing some "nots". These are

(a) The *contrapositive* of the statement: If not Q, then not P.
(b) The *converse* of the statement: If Q, then P.
(c) The *inverse* of the statement: If not P, then not Q.

The contrapositive of a statement is logically equivalent to it, that is, the contrapositive is true if and only if the original implication is true. This is a useful fact in constructing proofs. (See an example below.)

The truth of the converse and inverse, on the other hand, is not related to that of the original statement, though they are equivalent to one another. (Why?)

Exercise 1.3.1. Write the contrapositive, converse, and inverse of each of the following true statements. Do you agree that the contrapositive is true in each case? What about the converse and inverse?

(a) If $2n$ is an odd integer, then n is not an integer.
(b) If $m > 0$, then $m^2 > 0$.

1.3.2. If and only if. In mathematics we often encounter situations that both $P \Rightarrow Q$ and $Q \Rightarrow P$ are true. Then we write $P \Leftrightarrow Q$ and say that P and Q are equivalent. As before, there are several different ways of saying this in English. A popular one is to say: "P if and only if Q" or "P is necessary and sufficient condition for Q."

1.3.3. Quantifiers. Mathematical statements usually involve *quantifiers*, although they are not always made explicit. We write things like: "For every integer n, $n(n+1)$ is even." or "$n(n+1)$ is even whenever n is an integer."

Some statements may involve several nested quantifiers: "For every cubic polynomial p with real coefficients there exists a real number x such that $p(x) = 0$."

Notice that the order of quantifiers is important.

Exercise 1.3.2. Explain the difference in meaning between the statement just given and this one: "There exists a real number x such that for every cubic polynomial f, $f(x) = 0$."

There are a number of different ways to express in English both the *universal quantifier* (for every, for each, for all...) and the *existential quantifier* (there exists, there is at least one...). We will regard each of these phrases as having exactly the same meaning as each of the others in its category. The logical symbol for the universal quantifier is $\forall$ and for the existential quantifier $\exists$.

1.3.4. Negations. It is often necessary to form the *negation* of a given statement. This is the statement that is true if and only if the original statement is false. (Thus the negation of the negation is the original statement.) Forming the negation is straightforward, but can demand careful attention if the original statement has many parts. Here are some examples.

Statement: If today is Tuesday, then the Western Front is published today.
Negation: Today is Tuesday, and the Western Front is not published today.

Recall that an implication is false only when the "if part" is true and the "then part" is false. Thus the negation must be true exactly under those conditions.

Statement: Bob and Bill are Western students.
Negation: Bob is not a Western student or Bill is not a Western student.

Notice that the original statement becomes false as soon as one man fails to be a Western student. Notice also that the second statement is still true if neither Bob nor Bill is a Western student. "Or" is always used in this way in mathematics.

Statement: Bob or Bill is a Western student.
Negation: Bob and Bill are not Western students.

In the same way, "for every" and "there exists" are interchanged when forming a negation.

Statement: For every x in A, $f(x) > 5$.
Negation: There exists an x in A such that $f(x) \leq 5$.

Statement: There is a rational number r such that $r^2 = 2$.
Negation: For every rational number r, $r^2 \neq 2$.

Here are some more complicated examples.

Statement: For every cubic polynomial f, there exists a real number x such that $f(x) = 0$.
Negation: There exists a cubic polynomial f such that for every real number x, $f(x) \neq 0$.

Statement: There exists a real number x such that for every cubic polynomial f, $f(x) = 0$.
Negation: For every real number x there exists a cubic polynomial f such that $f(x) \neq 0$.

Statement: For every $n \geq N$ and every x in E, $|f_n(x) - f(x)| < 1$.
Negation: There exists an $n \geq N$ and an x in E such that $|f_n(x) - f(x)| \geq 1$.

Think carefully about what each statement means before deciding that you agree that the negations are correct.

Exercise 1.3.3. Form the negation of each statement. Express the negation so that the word "not" or "no" does not occur.

(a) For every $x > 1$, there exists a real number y such that $1 < y < x$.
(b) For every $x > 0$ there exists $y > 0$ such that $xy < 1$.
(c) For every $x > 0$ there exists $y > 0$ such that $xy > 1$.
(d) There exists $y > 0$ such that for every $x > 0$ we have $xy > 1$.
(e) There exists $x > 0$ such that for all $y > 0$ we have $xy \leq 1$.
(f) For every $y > 0$ there exists $x > 0$ such that $xy \leq 1$.

1.4. Proofs

Most of the time in this class we will be constructing proofs. Here are some simple examples illustrating different styles of proof.

The first is a *direct* proof. Here one simply begins with the hypotheses and any other usable facts and reasons until one reaches the conclusion.

Theorem 1.4.1. *The square of an odd integer has the form* $8k+1$ *for some integer* k.

Remark 1.4.2. Note that this is really an implication and could be rephrased: If n is an odd integer, then there is an integer k such that $n^2 = 8k + 1$.

Proof. First we need to rewrite the hypothesis in a more useful form. "n is odd" means that n is not divisible by 2, that is, if we try to do the division we'll get a quotient q and a remainder of 1. Equivalently, $n = 2q + 1$. Thus

$$n^2 = (2q + 1)^2 = 4q^2 + 4q + 1 = 4q(q + 1) + 1.$$

Now either q is even, $q = 2r$ for some integer r, or q is odd, $q = 2s+1$ for some integer s. In the first case,

$$n^2 = 8r(q+1) + 1.$$

In the second case $q + 1 = 2s + 2 = 2(s + 1)$ so that

$$n^2 = 8q(s+1) + 1.$$

Thus, we do have $n^2 = 8k + 1$ in either case; $k = r(q + 1)$ or $k = q(s + 1)$ as appropriate. □

The second very useful strategy to prove the implication $P \Rightarrow Q$ (P implies Q) is to prove its contrapositive $\neg Q \Rightarrow \neg P$ (not Q implies not P). As we noted earlier:

THE CONTRAPOSITIVE OF A STATEMENT IS LOGICALLY EQUIVALENT TO IT, THAT IS, THE CONTRAPOSITIVE IS TRUE IF AND ONLY IF THE ORIGINAL IMPLICATION IS TRUE. (This can be shown using truth tables.)

Remark 1.4.3. Whenever you work with an implication it is very useful to state its contrapositive as well. In fact, you should always write both direct implication and its contrapositive and then decide which one is easier to prove.

Theorem 1.4.4. *If n^2 is even, then n is even.*

PROOF. The contrapositive of this statement is (using the fact that every integer is either even or odd): if n is odd, then n^2 is odd. This has just been proved, since an integer of the form $8k + 1$ is certainly odd. □

The third strategy to prove an implication is a proof by contradiction. In a proof of $P \Rightarrow Q$ by contradiction one assumes both P and $\neg Q$ (not Q) and derives a contradiction. This establishes that $P \Rightarrow Q$ is true because the only way for this implication to be false is for P to be true and Q to be false.

Theorem 1.4.5. *$\sqrt{2}$ is irrational.*

PROOF. We can rephrase the theorem as the following implication: If $x^2 = 2$, then x is irrational.

Suppose that x is rational. Then $x = a/b$ for some integers a and b. We may assume that this fraction is in lowest terms, that is, that a and b have no common factor. Then $2 = x^2 = (a/b)^2$ or $2b^2 = a^2$. Thus a^2 is even. By the previous theorem, a is even, i.e., $a = 2c$ for some integer c. But then

$$2b^2 = (2c)^2 = 4c^2 \quad \text{or} \quad b^2 = 2c^2.$$

Thus b is also even. But this contradicts our choice of a and b as having no common factor. Thus assuming that x is rational has led to a contradiction and we can conclude that x must be irrational. □

The proof above is an example of a proof by contradiction. Very often proofs by contradiction are in fact direct proofs of the contrapositive in disguise.

THE DIRECT PROOF OF THE CONTRAPOSITIVE. We will prove the following implication: If x is rational, then $x^2 \neq 2$.

Let x be a rational number. Then there exist integers p and q which are not both even, such that $x = p/q$. Now we need to prove that $\big(p/q\big)^2 \neq 2$, or equivalently $p^2 \neq 2q^2$.

Consider two cases: Case 1: p is odd, and Case 2: p is even.
Case 1. Assume that p is odd. By Theorem 1.4.4, p^2 is odd. Since $2q^2$ is even we have $p^2 \neq 2q^2$.
Case 2. Assume that p is even. Then there exists an integer r such that $p = 2r$. Since not both q and p are even, q must be odd. Then q^2 is odd as well. Since $2r^2$ is even we have $2r^2 \neq q^2$. Consequently $4r^2 \neq 2q^2$. Since $4r^2 = (2r)^2 = p^2$, it follows that $p^2 \neq 2q^2$. $\square$

VERY OFTEN PROOFS BY CONTRADICTION ARE DISGUISED PROOFS OF THE CONTRAPOSITIVE. BEFORE YOU DO A PROOF BY CONTRADICTION YOU SHOULD TRY TO PROVE THE CONTRAPOSITIVE FIRST.

1.5. Sets

By a *set* A we mean a well-defined collection of objects such that it can be determined whether or not any particular object is an element of A. If a is an object in the set A we say that a is an *element* of A and write $a \in A$. The negation of $x \in A$ is $x \notin A$.

The *empty set* is the unique set which contains no elements. The empty set is denoted by the symbol $\emptyset$.

Generally, capital letters will be used to denote sets of objects and lower case letters to denote objects themselves. However, watch for deviations of this rule. We will be concerned mainly with sets of real numbers. The specially designed letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the following important sets of real numbers:

$\mathbb{N}$ denotes the set of all *natural numbers* (or *positive integers*),
$\mathbb{Z}$ denotes the set of all *integers*,
$\mathbb{Q}$ denotes the set of all *rational numbers*,
$\mathbb{R}$ denotes the set of all *real numbers*.

A set can be described by:

- a statement such as "Let A be the set of real solutions of the equation $x^2 - x = 0$."
- a listing of all the elements; for example $A = \{0, 1\}$.
- notation such as $A = \big\{x \in \mathbb{R} : x^2 = x\big\}$.

Notice the usage of the braces (or curly brackets) $\{$ and $\}$ in the above examples. They are used to delimit the sets. The number 0 is an important real number. However, $\{0\}$ is the set whose only element is 0.

The expression $\big\{x \in \mathbb{R} : x^2 = x\big\}$ is read as "the set of all real numbers x such that $x^2 = x$". Here the colon (:) is used as an abbreviation for the phrase "such that".

Definition 1.5.1. A set B is a *subset* of a set A if every element of B is also an element of A. In this case we write $B \subseteq A$ or $A \supseteq B$. Formally, $B \subseteq A$ if and only if $x \in B$ implies $x \in A$.

Since the implication $x \in \emptyset \Rightarrow x \in A$ is always true, the empty set is a subset of each set. Below is the set of all subsets of the set $\{-1, 0, 1\}$.

$$\big\{\emptyset, \{-1\}, \{0\}, \{1\}, \{-1, 0\}, \{-1, 1\}, \{0, 1\}, \{-1, 0, 1\}\big\}$$

Definition 1.5.2. Two sets A and B are *equal*, denoted $A = B$, if they contain precisely the same elements, that is, if $A \subseteq B$ and $B \subseteq A$.

Notice that the elements are not repeated in a set; for example $\{0, 1, 0\} = \{0, 1\}$. Also, the order in which elements are listed is not important: $\{3, 2, 1\} = \{1, 2, 3\}$.

Remark 1.5.3. Equality is allowed in the definition of a subset, that is, a set is a subset of itself. If we wish to exclude this possibility we say B is a *proper subset* of A and we write $B \subsetneq A$ or $B \subset A$. Formally, $B \subsetneq A$ if and only if $x \in B$ implies $x \in A$ and there exists $a \in A$ such that $a \notin B$.

The negation of $B \subseteq A$ is denoted by $B \not\subseteq A$. Formally, $B \not\subseteq A$ if and only if there exists $b \in B$ such that $b \notin A$.

Definition 1.5.4. The *union* of A and B is the set of all x such that x is an element of A or x is an element of B. It is denoted $A \cup B$. Thus

$$A \cup B = \big\{x \,:\, x \in A \ \text{ or } \ x \in B\big\}.$$

Remark 1.5.5. The conjunction "or" in mathematics is always in an inclusive sense, that is, it is allowed in the definition that x belong to both A and B. For example, $\{0, 1, 2, 3\} \cup \{2, 3, 4, 5\} = \{0, 1, 2, 3, 4, 5\}$.

Definition 1.5.6. The *intersection* of A and B is the set of all x such that x is an element of A and x is an element of B. It is denoted $A \cap B$. Thus

$$A \cap B = \big\{x \,:\, x \in A \quad \text{and} \quad x \in B\big\}.$$

Two sets A and B are said to be *disjoint* if their intersection is the empty set, i.e. if $A \cap B = \emptyset$.

Definition 1.5.7. The *difference* between the sets A and B is the set of all x such that x is an element of A and x is not an element of B. It is denoted $A \setminus B$. Thus

$$A \setminus B = \big\{x \,:\, x \in A \ \text{ and } \ x \notin B\big\}.$$

Definition 1.5.8. An *ordered pair* is a collection of two not necessarily distinct elements, one of which is distinguished as the first coordinate (or first entry) and the other as the second coordinate (second entry). The common notation for an ordered pair with first coordinate a and second coordinate b is (a, b).

Remark 1.5.9. The ordered pairs $(0, 1)$ and $(1, 0)$ are different since their first entries are different. The ordered pairs $(0, 0)$ and $(0, 1)$ are different since their second entries are different. In general, $(a, b) = (x, y)$ if and only if $a = x$ and $b = y$.

Notice the usage of the round brackets (and) in the definition of an ordered pair. Please distinguish between $\{0, 1\}$ and $(0, 1)$: $\{0, 1\}$ is a set with two elements, $(0, 1)$ is an ordered pair, an object defined by Definition 1.5.8.

Definition 1.5.10. The *Cartesian product* (or *direct product*) of two sets A and B, denoted $A \times B$, is the set of all possible ordered pairs whose first entry is a member of A and whose second entry is a member of B:

$$A \times B = \big\{(a, b) \ : \ a \in A \ \text{ and } \ b \in B\big\}.$$

The main example of a Cartesian product is $\mathbb{R} \times \mathbb{R}$ which provides a coordinate system for the plane.

Example 1.5.11. Let $A = \{1, 2, 3, 4\}$ and let $C = \{\mathtt{R}, \mathtt{G}, \mathtt{B}\}$ be the set of primary colors where R stands for red, G for green, and B for blue. Then

$$A \times C = \big\{(1,\mathtt{R}), (1,\mathtt{G}), (1,\mathtt{B}), (2,\mathtt{R}), (2,\mathtt{G}), (2,\mathtt{B}), \\ (3,\mathtt{R}), (3,\mathtt{G}), (3,\mathtt{B}), (4,\mathtt{R}), (4,\mathtt{G}), (4,\mathtt{B})\big\}.$$

Ideally, mathematical terminology and notation should be completely free of ambiguities. We strive for the absolute certainty. However, very soon we will introduce the concept of an open interval and for this concept we will use the same notation as for an ordered pair. It should be clear from the context what is meant. Whenever you are uncertain look for the resolution of the uncertainty.

We conclude this section with a remark about families of sets. In this class we mostly talk about sets of real numbers. Sometimes we will talk about sets whose elements are also sets. It is customary to use the word "*family*" instead of "set" when we talk about sets of sets; see examples in Section 2.3.

For any nonempty family of sets we can define the concepts of union and intersection. Let $\mathcal{A}$ be a nonempty family of sets. We define the intersection of the family $\mathcal{A}$ to be

$$\bigcap \big\{A : A \in \mathcal{A}\big\} := \big\{x : x \in A \ \text{ for all } \ A \in \mathcal{A}\big\}.$$

We define the union of the family $\mathcal{A}$ to be

$$\bigcup \big\{A : A \in \mathcal{A}\big\} := \big\{x : x \in A \ \text{ for some } \ A \in \mathcal{A}\big\}$$

1.6. Functions

Let A and B be nonempty sets. A *function* from A to B is a rule f which assigns a unique element of B to each element of A.

The set A is called the *domain* of the function. We denote by $f(x)$ the element of B which is assigned to a particular $x \in A$. This element is called a value of f at x, or image of x under f.

As a simple example we can define the identity function on a set A, $\mathrm{id}_A : A \to A$, by $\mathrm{id}_A(x) = x$ for all $x \in A$.

A weakness of the above definition of a function is that it relies on the undefined concept of a "rule". It is not clear what constitutes a valid rule defining a function. To overcome this weakness we identify a function f with its graph G_f which is a subset of the cartesian product $A \times B$:

$$G_f = \big\{\big(x, f(x)\big) : x \in A\big\}.$$

and we require that for each x in A there is at most one pair (x, y) in this subset. The formal definition of a function from A to B is given in terms of subsets of $A \times B$.

Definition 1.6.1. A *function* from A into B is a subset G_f of the Cartesian product $A \times B$ such that

(i) for every $x \in A$ there exists $y \in B$ such that $(x, y) \in G_f$;
(ii) if $(x, y), (x, z) \in G_f$, then $y = z$.

Consider the sets A and C given in Example 1.5.11. The subset

$$\big\{(1,\mathtt{G}),(2,\mathtt{R}),(3,\mathtt{G}),(4,\mathtt{B})\big\}.$$

of $A \times C$ is a function in the sense of Definition 1.6.1. In the traditional notation this function is given by $f(1) = \mathtt{G}, f(2) = \mathtt{R}, f(3) = \mathtt{G}, f(4) = \mathtt{B}$. In contrast, the subset

$$\big\{(1,\mathtt{B}),(2,\mathtt{G}),(2,\mathtt{R}),(3,\mathtt{R}),(4,\mathtt{G})\big\}$$

is not a function since $(2,\mathtt{G}),(2,\mathtt{R})$ are in the set and $\mathtt{G} \neq \mathtt{R}$. Hence (ii) in Definition 1.6.1 does not hold for this set.

For small sets A and B we can list all the functions from A to B.

Example 1.6.2. Let $A = \{0,1\}$ and let $C = \{\mathtt{R},\mathtt{G},\mathtt{B}\}$. The following is the list of all functions from A to C.

$$\big\{(0,\mathtt{R}),(1,\mathtt{R})\big\},\quad \big\{(0,\mathtt{R}),(1,\mathtt{G})\big\},\quad \big\{(0,\mathtt{R}),(1,\mathtt{B})\big\},\quad \big\{(0,\mathtt{G}),(1,\mathtt{R})\big\},\quad \big\{(0,\mathtt{G}),(1,\mathtt{G})\big\}$$
$$\big\{(0,\mathtt{G}),(1,\mathtt{B})\big\},\quad \big\{(0,\mathtt{B}),(1,\mathtt{R})\big\},\quad \big\{(0,\mathtt{B}),(1,\mathtt{G})\big\},\quad \big\{(0,\mathtt{B}),(1,\mathtt{B})\big\}.$$

In the rest of these notes we will use the informal definition of a function. The symbol $f : A \to B$ stands for a function from A to B. If we want to emphasize the rule that defines f we write $f : x \mapsto f(x),\ x \in A$. For example, $x \mapsto x^2,\ x \in \mathbb{R}$, denotes the square function defined on $\mathbb{R}$ without giving this function a specific name.

The set $\big\{f(x) : x \in A\big\}$ is the *range* of f. Formally, y is in the range of f if and only if there exists $x \in A$ such that $y = f(x)$.

A function $f : A \to B$ is *one-to-one* (or an *injection*) if distinct elements of A have distinct images in B, i.e., if for all $x, y \in A$, $x \neq y$ implies $f(x) \neq f(y)$. Notice that the contrapositive of the last implication is: for all $x,\ y \in A$, $f(x) = f(y)$ implies $x = y$. To prove that a function $f : A \to B$ is not one-to-one we have to find $x_1, x_2 \in A$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

There are only three functions listed in Example 1.6.2 which are not one-to-one. Find them!

The function $x \mapsto x^2,\ x \in \mathbb{R}$, is not one-to-one since -1 and 1 are in the domain of this function and $-1 \neq 1$ and $1 = (-1)^2 = 1^2$. However, with $A = \{x \in \mathbb{R} : x \geq 0\}$, the function $x \mapsto x^2,\ x \in A$, is one-to-one. This will be proved in the next chapter.

A function $f : A \to B$ is *onto* (or a *surjection*) if for every point $y \in B$ there is at least one point $x \in A$ such that $f(x) = y$. Another way of saying that $f : A \to B$ is onto B is to say that the range of f is the whole of B. To prove that $f : A \to B$ is not onto we have to prove that there exists $b \in B$ such that for all $x \in A$ we have $f(x) \neq b$.

Let $A = \{x \in \mathbb{R} : x \geq 0\}$ and $s(x) = x^2, x \in \mathbb{R}$. Then $s : \mathbb{R} \to A$ is a surjection. To prove this we have to prove that for every $a \geq 0$ there exists $x \in \mathbb{R}$ such that $x^2 = a$. The case $a = 0$ is easy; we can take $x = 0$. The case $a > 0$ will be discussed at the end of the next section.

It is interesting to note that with $B = \{x \in \mathbb{Q} : x \geq 0\}$ the function $s : \mathbb{Q} \to B$ is not a surjection. This was essentially proved in Theorem 1.4.5. In the direct proof of the contrapositive of this theorem we proved that $x^2 \neq 2$ for every $x \in \mathbb{Q}$. Since $2 \in B$, this proves that $s : \mathbb{Q} \to B$ is not a surjection.

A function $f : A \to B$ which is both one-to-one and onto is called *bijection*.

Let $f : A \to B$ and $g : C \to D$ be given functions. Assume that the range of f is contained in the domain of g. Then we can define the function $h : A \to D$ by

$$h(x) = g\bigl(f(x)\bigr), \qquad x \in A.$$

The function h is called a composition of f and g and it is denoted by $g \circ f$.

Exercise 1.6.3. Let A and B be nonempty sets. Let $f : A \to B$ be a given function. Prove f is a bijection if and only if there exists a function $h : B \to A$ such that $h \circ f = \mathrm{id}_A$ and $f \circ h = \mathrm{id}_B$.

Solution. Assume that $f : A \to B$ is a bijection. Then for every $b \in B$ there exists unique $a \in A$ such that $f(a) = b$. Define the function $h : B \to A$ by $h(y) = x$. Let $x \in A$ be arbitrary and let $y = f(x)$. Then, by the definition of h, $h(y) = x$ and $h(f(x)) = x$. Since $x \in A$ was arbitrary we proved that $h \circ f = \mathrm{id}_A$. Let $v \in B$ be arbitrary and let $v = f(u)$. Then, by the definition of h, $h(v) = u$ and $f(h(v)) = v$. Since $v \in B$ was arbitrary we proved that $f \circ h = \mathrm{id}_B$.

To prove the converse, assume that there exists a function $h : B \to A$ such that $h \circ f = \mathrm{id}_A$ and $f \circ h = \mathrm{id}_B$. To prove that f is a surjection, let $b \in B$ be arbitrary. Set $a = h(b)$. Than $f(a) = f(h(b)) = \mathrm{id}_B(b) = b$. Hence f is a surjection. To prove that f is an injection let $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$. Since $f(a_1) = f(a_2) \in B$ and since $h : B \to A$ is a function, we have $h(f(a_1)) = h(f(a_2))$. Since $h \circ f = \mathrm{id}_A$ we have $h(f(a_1)) = \mathrm{id}_A(a_1) = a_1$ and $h(f(a_2)) = \mathrm{id}_A(a_2) = a_2$. Thus $a_1 = a_2$. This proves that f is an injection. □

Let $f : A \to B$ be a function. The function $h : B \to A$ such that $h \circ f = \mathrm{id}_A$ and $f \circ h = \mathrm{id}_B$ is called the *inverse function* of f. It is denoted by f^{-1}. Thus $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$. A function f which possesses an inverse is said to be *invertible*. By Exercise 1.6.3 a function f is invertible if and only if it is a bijection.

Exercise 1.6.4. Let A, B and C be nonempty sets. Let $f : A \to B$ and $g : B \to C$ be injections. Prove that $g \circ f : A \to C$ is an injection.

Exercise 1.6.5. Let A, B and C be nonempty sets. Let $f : A \to B$ and $g : B \to C$ be surjections. Prove that $g \circ f : A \to C$ is a surjection.

Solution. To prove that $g \circ f : A \to C$ is a surjection we have to prove that for each $c \in C$ there exists $a \in A$ such that $g(f(a)) = c$. Let $c \in C$ be arbitrary. Then, since $g : B \to C$ is a surjection, there exists $b \in B$ such that $g(b) = c$. Since $b \in B$ and since $f : A \to B$ is a surjection, there exists $a \in A$ such that $f(a) = b$. Now it is easy to show that $g(f(a)) = g(b) = c$. □

Exercise 1.6.6. Let A, B and C be nonempty sets. Let $f : A \to B$ and $g : B \to C$ be bijections. Prove that $g \circ f : A \to C$ is a bijection. Prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

We conclude this section with a negation exercise.

Exercise 1.6.7. Formulate the negation of the following statement.

Statement. *Let A and B be nonempty sets. There exists a surjection $f : A \to B$.*

Solution. The negation is: For an arbitrary function $g : A \to B$, g is not a surjection. But the statement "g is not a surjection" is itself a negation which means: There exists $b \in B$ such that for all $x \in A$ we have $g(x) \neq b$. Hence the negation of the given claim is:

For an arbitrary function $g : A \to B$ there exists $b \in B$ such that for all $x \in A$ we have $g(x) \neq b$. Symbolically this can be written as

$$\forall\ g : A \to B \quad \exists\ b \in B \quad \text{such that} \quad \forall\ x \in A \quad g(x) \neq b.$$

Sometimes the set of all functions defined on A with the values in B is denoted by B^A. With this notation the last statement can be written nicer as

$$\forall\ g \in B^A \quad \exists\ b \in B \quad \text{such that} \quad \forall\ x \in A \quad g(x) \neq b.$$

It is important to note that b in this statement depends on g. In a proof the last statement one would start from an arbitrary g and then try to construct $b \in B$ with the desired property. □

1.7. Four basic ingredients of a Proof

Since in this course you will be writing your own proofs and studying proofs of others, we conclude this chapter with four basic ingredients of a Proof.

A proof should contain ingredients which answer the following four questions:

- What is being assumed?
- What is being proved?
- What are the tools that are being used?
- Why is it legitimate to use those tools?

Sometimes the presence of these ingredients in a proof is implicit. But, it should always be easy to identify them.

These four questions are a good starting point when you critically evaluate your own proofs or when you comment on the proofs of others.

CHAPTER 2

The Set $\mathbb{R}$ of real numbers

All concepts that we will study in this course have their roots in the set of real numbers. We assume that you are familiar with some basic properties of the real numbers $\mathbb{R}$ and of the subsets $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ of $\mathbb{R}$. However, in order to clarify exactly what we need to know about $\mathbb{R}$, we set down its basic properties (called axioms) and some of their consequences.

2.1. Axioms of a field

The following are the basic properties (axioms) of $\mathbb{R}$ that relate to addition and multiplication in $\mathbb{R}$:

Axiom 1 (A0)**.** If $a, b \in \mathbb{R}$, then the sum $a + b$ is uniquely defined element in $\mathbb{R}$. That is, there exists a function $+$ (called "plus") defined on $\mathbb{R} \times \mathbb{R}$ and with the values in $\mathbb{R}$.

Axiom 2 (A1)**.** $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{R}$.

Axiom 3 (A2)**.** $a + b = b + a$ for all $a, b \in \mathbb{R}$.

Axiom 4 (A3)**.** There exists an element 0 in $\mathbb{R}$ such that $0 + a = a + 0 = a$ for all $a \in \mathbb{R}$.

Axiom 5 (A4)**.** If $a \in \mathbb{R}$, then the equation $a + x = 0$ has a solution $-a \in \mathbb{R}$.

Axiom 6 (M0)**.** If $a, b \in \mathbb{R}$, then the product $a \cdot b$ (usually denoted by ab) is uniquely defined number in $\mathbb{R}$. That is, there exists a function $\cdot$ (called "times") defined on $\mathbb{R} \times \mathbb{R}$ and with the values in $\mathbb{R}$.

Axiom 7 (M1)**.** $a(bc) = (ab)c$ for all $a, b, c \in \mathbb{R}$.

Axiom 8 (M2)**.** $ab = ba$ for all $a, b \in \mathbb{R}$.

Axiom 9 (M3)**.** There exists an element 1 in $\mathbb{R}$ such that $1 \neq 0$ and $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{R}$.

Axiom 10 (M4)**.** If $a \in \mathbb{R}$ and $a \neq 0$, then the equation $a \cdot x = 1$ has a solution $a^{-1} = \dfrac{1}{a}$ in $\mathbb{R}$.

Axiom 11 (DL)**.** $a(b + c) = ab + ac$ for all $a, b, c \in \mathbb{R}$.

Remark 2.1.1. Notice that the only specific real numbers mentioned in the axioms are 0 and 1. You can verify that the set $\{0, 1\}$ with the functions $\oplus$ (a special addition instead of $+$) and $\odot$ (a special multiplication instead of $\cdot$) defined by

$$0 \oplus 0 = 1 \oplus 1 = 0, \ 0 \oplus 1 = 1 \oplus 0 = 1 \quad \text{and} \quad 0 \odot 0 = 0 \odot 1 = 1 \odot 0 = 0, \ 1 \odot 1 = 1.$$

satisfy all Axioms 1 through 11. Hence, we need more axioms to describe the set of real numbers.

Axioms A1 and M1 are called *associative laws* and Axioms A2 and M2 are *commutative laws*. Axiom DL is the *distributive law*; this law justifies "factorization" and "multiplying out" in algebra. A triple of a set, "plus" and "times" functions which satisfies Axioms 1 through 11 is called a *field*. The basic algebraic properties of $\mathbb{R}$ can be proved solely on the basis of the field axioms. We illustrate this claim by the following exercise.

Exercise 2.1.2. Let $a, b, c \in \mathbb{R}$. Prove the following statements.

(a) If $a + c = b + c$, then $a = b$.
(b) $a \cdot 0 = 0$ for all $a \in \mathbb{R}$.
(c) $-a = a$ if and only if $a = 0$.
(d) $-(-a) = a$ for all $a \in \mathbb{R}$.
(e) $(-a)b = -(ab)$ for all $a, b \in \mathbb{R}$.
(f) $(-a)(-b) = ab$ for all $a, b \in \mathbb{R}$.
(g) If $a \neq 0$, then $\bigl(a^{-1}\bigr)^{-1} = a$.
(h) If $ac = bc$ and $c \neq 0$, then $a = b$.
(i) $ab = 0$ if and only if $a = 0$ or $b = 0$.
(j) If $a \neq 0$ and $b \neq 0$, then $(ab)^{-1} = a^{-1}b^{-1}$.

Remark 2.1.3. We will prove (a), (e) and a part of (i) below. Others you can do as exercise. One of the statements in Exercise 2.1.2 cannot be proved using Axioms 1 through 11. To prove that particular property we will need results from Section 2.2.

Solution. (a) Assume that $a+c = b+c$. By Axiom 1 adding any number x to both sides of the equality leads to $(a+c)+x = (b+c)+x$. It follows from Axiom 2 that $a+(c+x) = b+(c+x)$. By Axiom 5 there exists an element $-c \in \mathbb{R}$ such that $c + (-c) = 0$. Choose $x = -c$. Then $a = a + 0 = a + (c + (-c)) = b + (c + (-c)) = b + 0 = b$.

(e) Let $a, b \in \mathbb{R}$. Then, by Axiom 5 there exists $-a \in \mathbb{R}$ such that $a+(-a) = 0$. By Axiom 6 it follows that $\bigl(a + (-a)\bigr)b = 0 \cdot b$. By Axiom 11 and part (b) of this exercise, it follows that $ab + (-a)b = 0$. Since $ab \in \mathbb{R}$, by Axiom 5 there exists $-(ab) \in \mathbb{R}$ such that $ab + (-(ab)) = 0$. Using Axiom 3 we conclude that $(-a)b+ab = -(ab)+ab$. By part (a) of this proof we conclude that $(-a)b = -(ab)$.

We prove "only if" part of (i). That is, we prove the implication:

$$ab = 0 \quad \text{implies} \quad a = 0 \quad \text{or} \quad b = 0. \tag{2.1.1}$$

Assume that $ab = 0$. Consider two cases: Case 1: $a = 0$ and Case 2: $a \neq 0$.
Case 1. In this case the implication (2.1.1) is true and there is nothing to prove.
Case 2. Since in this case we assume that $a \neq 0$, by Axiom 10 there exists an element $a^{-1} \in \mathbb{R}$ such that $aa^{-1} = 1$. Multiplying both sides of $ab = 0$ by a^{-1} we get $(ab)a^{-1} = 0 \cdot a^{-1}$. Therefore

$$b = b \cdot 1 = b(aa^{-1}) = (ba)a^{-1} = (ab)a^{-1} = 0 \cdot a^{-1} = 0. \qquad \square$$

Remark 2.1.4. Let $a, b \in \mathbb{R}$. Instead of $a + (-b)$ we write $a - b$ and we write $\dfrac{a}{b}$ or a/b instead of ab^{-1}.

2.2. Axioms of order in a field

The set $\mathbb{R}$ also has an order structure $<$ satisfying the following axioms.

Axiom 12 (O1)**.** Given any $a, b \in \mathbb{R}$, exactly one of the following three statements is true: $a < b$, $a = b$, or $b < a$.

Axiom 13 (O2)**.** Given any $a, b, c \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$.

Axiom 14 (O3)**.** Given any $a, b, c \in \mathbb{R}$, if $a < b$ then $a + c < b + c$.

Axiom 15 (O4)**.** Given any $a, b, c \in \mathbb{R}$, if $a < b$ and $0 < c$, then $ac < bc$.

Axiom O2 is called the *transitive law*. A field with an order satisfying Axioms O1 through O4 is called an *ordered field*.

The notation $a \leq b$ stands for the statement: $a < b$ or $a = b$.

Definition 2.2.1. A number $x \in \mathbb{R}$ is *positive* if $x > 0$. A number $x \in \mathbb{R}$ is *negative* if $x < 0$.

Exercise 2.2.2. Prove the following statements for $a, b, c \in \mathbb{R}$.
(a) If $a < b$ then $-b < -a$.
(b) If $a < b$ and $c < 0$, then $b\,c < a\,c$.
(c) Assume $a > 0$ and $b \neq 0$. Prove that $b > 0$ if and only if $ab > 0$.
(d) If $a \neq 0$, then $0 < aa$.
(e) $0 < 1$.
(f) If $a > 0$, then $\dfrac{1}{a} > 0$.
(g) If $0 < a < b$, then $0 < \dfrac{1}{b} < \dfrac{1}{a}$.

SOLUTION. (a) Assume $a < b$. By Axiom 14 we have $a+(-b) < b+(-b)$. Thus $(-b)+a < 0$. Using Axiom 14 again, we conclude that $\big((-b)+a\big)+(-a) < 0+(-a)$, and consequently $-b < -a$.

Do (b) as an exercise.

Now we prove (c). Assume $a > 0$ and $b \neq 0$. This assumption is used throughout this part of the proof. Since $a \neq 0$ and $b \neq 0$, by Exercise 2.1.2 (i) it follows that $ab \neq 0$. The implication: "If $b > 0$, then $ab > 0$." is a special case of Axiom 15. Next we deal with the implication "If $ab > 0$, then $b > 0$." It turns out that the contrapositive is easier to prove. The negation of $b > 0$ is $b \leq 0$. But, it is assumed that $b \neq 0$. Thus, with this assumption, the negation of $b > 0$ is $b < 0$. Similarly, the negation of $ab > 0$ is $ab < 0$. Hence the contrapositive of "If $ab > 0$, then $b > 0$." is "If $b < 0$, then $ab < 0$." The last implication follows directly from part (b). This completes the proof of (c).

(d) Consider two cases: $a > 0$ and $a < 0$. If $a > 0$, then (c) implies that $a^2 > 0$. If $a < 0$, then, by (a), $-0 < -a$, and since $-0 = 0$ we have $-a > 0$. By the first part of this proof, we conclude that $(-a)(-a) > 0$. By part (f) of Exercise 2.1.2 we have $(-a)(-a) = a\,a$. Therefore $aa > 0$ for all $a \neq 0$.

Do (e) as an exercise.

To prove (f) we assume $a > 0$. By Axiom 10, $a\,\dfrac{1}{a} = 1$. By Axiom 9, $1 \neq 0$. Hence, $a\,\dfrac{1}{a} \neq 0$. By Exercise 2.1.2 (i) $\dfrac{1}{a} \neq 0$ and by (e) $1 > 0$. Now we can apply the "if" part of (c). (Take $b = 1/a$ in (c).) We conclude that $a\,\dfrac{1}{a} = 1 > 0$ implies $\dfrac{1}{a} > 0$. This proves (f).

Do (g) as an exercise. $\square$

Definition 2.2.3. We define the following eight numbers

$$2 = 1 + 1, \qquad 3 = 2 + 1, \qquad 4 = 3 + 1, \qquad 5 = 4 + 1,$$
$$6 = 5 + 1, \qquad 7 = 6 + 1, \qquad 8 = 7 + 1, \qquad 9 = 8 + 1.$$

The numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ are called *digits*.

In the preceding definition we implied that the digits are distinct numbers. The next exercise justifies this claim.

Exercise 2.2.4. Prove the inequalities:

$$0 < 1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$$

Exercise 2.2.5. Let $a, b \in \mathbb{R}$. If $a < b$, then there exists $c \in \mathbb{R}$ such that $a < c < b$.

The following four exercises deal with squares of real numbers. As usual, for $a \in \mathbb{R}$, a product aa is called a *square* and it is denoted by a^2.

Exercise 2.2.6. Let $a \in \mathbb{R}$. Prove that the equation $x^2 = a$, has at most two solutions in $\mathbb{R}$.

Solution. Consider the set

$$S = \bigl\{x \in \mathbb{R} : x^2 = a\bigr\}.$$

If $S = \emptyset$, then the statement is true. Now assume that $S \neq \emptyset$ and let $b \in S$. From $b \in S$, we deduce that $b \in \mathbb{R}$ and $b^2 = a$. Since $b \in \mathbb{R}$, $-b \in \mathbb{R}$. Next we will prove

$$S = \bigl\{b, -b\bigr\}. \tag{2.2.1}$$

Let $c \in S$. Then $c^2 = a$, and therefore $c^2 = b^2$. Consequently, $c^2 - b^2 = 0$. Using Axioms 2 through 11 and properties in Exercise 2.1.2 we can prove that $(c - b)(c + b) = c^2 - b^2$. Therefore $(c - b)(c + b) = c^2 - b^2 = 0$. Exercise 2.1.2 (i) implies that $c - b = 0$ or $c + b = 0$. Thus $c = b$ or $c = -b$. This proves

$$S \subseteq \{b, -b\}. \tag{2.2.2}$$

Next we prove $\{b, -b\} \subseteq S$. By assumption $b \in S$. Since $(-b)^2 = b^2$, we have $(-b)^2 = a$. Hence $-b \in S$. Therefore

$$\{b, -b\} \subseteq S. \tag{2.2.3}$$

Relations (2.2.2) and (2.2.3) imply equality (2.2.1). Since the set $\{b, -b\}$ has at most two elements the statement is proved. □

Exercise 2.2.7. Let $0 \leq x, y$. Prove that $x < y$ if and only if $x^2 < y^2$.

Exercise 2.2.8. If $\alpha > 1$ and $\alpha > x^2$, then $\alpha > x$.

Exercise 2.2.9. If $s \neq t$, then $(s + t)^2 > 4st$.

Exercise 2.2.10. Let $a, b, c, d \in \mathbb{R}$.

(i) Prove or disprove the statement: If $a < b$ and $c < d$, then $a - c < b - d$.
(ii) If you disproved the statement in (i), change the assumptions about c and d to make a correct statement. Prove your new statement.

Exercise 2.2.11. Let $\alpha \in \mathbb{R}$. Prove that $\alpha < x$, $\forall\, x > 0$, implies $\alpha \leq 0$.

The properties of real numbers proved in this and the previous section are essential. Many of them are truly elementary (although sometimes hard to prove) and you can (and I will) use such properties in proofs without any justification. But, when you are using more subtle properties (like ones in Exercises 2.2.7, 2.2.8, or 2.2.10) you should state explicitly which property you are using and explain informally why it is true.

2.3. Intervals

Exercise 2.2.5 justifies the following definition.

Definition 2.3.1. Let a and b be real numbers such that $a < b$. We will use the following notation and terminology:

$$[a, b] := \{x \in \mathbb{R} : a \le x \le b\} \quad \text{is called } \textit{a closed interval},$$
$$(a, b) := \{x \in \mathbb{R} : a < x < b\} \quad \text{is called } \textit{an open interval},$$
$$[a, b) := \{x \in \mathbb{R} : a \le x < b\} \quad \text{is called } \textit{a half-open interval},$$
$$(a, b] := \{x \in \mathbb{R} : a < x \le b\} \quad \text{is called } \textit{a half-open interval}.$$

We also define four types of unbounded intervals:

$$[a, +\infty) := \{x \in \mathbb{R} : a \le x\} \quad \text{is called } \textit{a closed unbounded interval},$$
$$(a, +\infty) := \{x \in \mathbb{R} : a < x\} \quad \text{is called } \textit{an open unbounded interval}$$
$$(-\infty, b] := \{x \in \mathbb{R} : x \le b\} \quad \text{is called } \textit{an unbounded closed interval},$$
$$(-\infty, b) := \{x \in \mathbb{R} : x < b\} \quad \text{is called } \textit{an unbounded open interval},$$

Geometric illustrations of these intervals are given below.

Figure 1. A closed interval

Figure 2. An open interval

Figure 3. A half-open interval

Figure 4. A half-open interval

Figure 5. A closed infinite interval

Figure 6. An open infinite interval

Figure 7. An infinite closed interval

Figure 8. An infinite open interval

Remark 2.3.2. The infinity symbols $-\infty$ and $+\infty$ are used to indicate that the set is unbounded in the negative ($-\infty$) or positive ($+\infty$) direction of the real number line. The symbols $-\infty$ and $+\infty$ are just symbols; they are not real numbers. Therefore we always exclude them as endpoints by using parentheses.

We conclude this section with few exercises about families of intervals.

Exercise 2.3.3. Let $a \in \mathbb{R}$. Prove that

$$\bigcap \bigl\{ (a-u, a+u) \,:\, u > 0 \bigr\} = \{a\}.$$

Exercise 2.3.4. Let $a, b \in \mathbb{R}$ and $a < b$. Prove that

$$\bigcap \bigl\{ (a, b+u) \,:\, u > 0 \bigr\} = (a, b].$$

Exercise 2.3.5. Let $a, b \in \mathbb{R}$ and $a < b$. Prove that

$$\bigcap \Bigl\{ (a-u, b+u) \,:\, u > 0 \Bigr\} = [a, b].$$

SOLUTION. Denote by A the intersection in the equality and assume $x \in A$. Then, by the definition of intersection, $x \in (a-u, b+u)$ for all $u > 0$. By the definition of an open interval, $a - u < x$ and $x < b + u$ for all $u > 0$. Hence, $a - x < u$ and $x - b < u$ for all $u > 0$. By Exercise 2.2.11 we have $a - x \leq 0$ and $x - b \leq 0$, that is, $a \leq x$ and $x \leq b$. By the definition of a closed interval $x \in [a, b]$. This proves $A \subseteq [a, b]$.

Now assume that $x \in [a, b]$. Then, $a - x \leq 0$ and $x - b \leq 0$. Let $u > 0$ be arbitrary. By the transitivity of the order in $\mathbb{R}$, $a - x < u$ and $x - b < u$ for all $u > 0$. Hence, $a - u < x$ and $x < b + u$ for all $u > 0$. Consequently, $x \in (a-u, b+u)$ for all $u > 0$. Therefore, $x \in A$. This proves $[a, b] \subseteq A$.

Since we proved both $A \subseteq [a, b]$ and $[a, b] \subseteq A$, the equality $A = [a, b]$ is proved. □

Exercise 2.3.6. Let $a, b \in \mathbb{R}$ and $a < b$. Prove that

$$\bigcup \Bigl\{ [a+u, b) \,:\, 0 < u < b - a \Bigr\} = (a, b).$$

Exercise 2.3.7. Let $a, b \in \mathbb{R}$ and $a < b$. Prove that

$$\bigcup \Bigl\{ [a+u, b-u] \,:\, 0 < u < \frac{b-a}{2} \Bigr\} = (a, b).$$

2.4. Bounded sets. Minimum and Maximum

Definition 2.4.1. Let A be a nonempty subset of $\mathbb{R}$. If there exists $b \in \mathbb{R}$ such that

$$x \leq b \quad \text{for all} \quad x \in A, \tag{2.4.1}$$

then A is said to be *bounded above*. A number b satisfying (2.4.1) is called an *upper bound* of A.

Similarly we define:

Definition 2.4.2. Let A be a nonempty subset of $\mathbb{R}$. If there exists $a \in \mathbb{R}$ such that

$$a \leq x \quad \text{for all} \quad x \in A, \tag{2.4.2}$$

then A is said to be *bounded below*. A number a satisfying (2.4.2) is called a *lower bound* of A.

Definition 2.4.3. A nonempty subset of $\mathbb{R}$ which is both bounded above and bounded below is said to be *bounded*.

Exercise 2.4.4. Let A be a nonempty subset of $\mathbb{R}$. Prove that A is bounded if and only if there exists $K > 0$ such that $-K \leq x \leq K$ for all $x \in A$.

Exercise 2.4.5. Let A be a nonempty subset of $\mathbb{R}$. Prove that A is bounded if and only if there exist $a, b \in \mathbb{R}$, such that $a < b$ and $A \subseteq [a, b]$.

Exercise 2.4.6. Prove that $\{x \in \mathbb{R} : x^2 < 2\}$ is a bounded set.

Exercise 2.4.7. Let A and B be bounded above subsets of $\mathbb{R}$. Prove that $A \cup B$ is bounded above.

Next we introduce the definitions of the minimum and the maximum.

Definition 2.4.8. Let A be a nonempty subset of $\mathbb{R}$. A number $a \in \mathbb{R}$ is a *minimum* of A if it has the following two properties:

(i) $a \leq x$ for all $x \in A$; (ii) $a \in A$.

The minimum of A (if it exists) is denoted by $\min A$.

Definition 2.4.9. Let A be a nonempty subset of $\mathbb{R}$. A number $b \in \mathbb{R}$ is a *maximum* of A if it has the following two properties:

(i) $x \leq b$ for all $x \in A$; (ii) $b \in A$.

The maximum of A (if it exists) is denoted by $\max A$.

Exercise 2.4.10. Let $x, y \in \mathbb{R}$. Prove that the set $A = \{x, y\}$ has a minimum and a maximum.

Remark 2.4.11. What does it mean for a nonempty subset of $\mathbb{R}$ not to have a minimum? To answer this question we first restate Definition 2.4.8 as follows. A nonempty set A has a minimum if

$$\exists\, a \in A \quad \text{such that} \quad \forall\, x \in A \quad \text{we have} \quad x \geq a. \tag{2.4.3}$$

Next we formulate the negation of the statement (2.4.3):

$$\forall\, a \in A \quad \exists\, x \in A \quad \text{such that} \quad x < a. \tag{2.4.4}$$

Notice that the number x in (2.4.4) depends on a. Sometimes it is useful to emphasize this dependence by writing $x(a)$. A more precise version of the negation is:

$$\forall\, a \in A \quad \exists\, x(a) \in A \quad \text{such that} \quad x(a) < a.$$

Exercise 2.4.12. Prove that the set of all positive numbers does not have a minimum.

Exercise 2.4.13. Give examples of subsets of $\mathbb{R}$ such that:

(a) A set has neither a minimum nor a maximum.
(b) A set has a minimum but not a maximum.
(c) A set has a minimum and a maximum.

2.5. Three functions: the unit step, the sign and the absolute value

There are only two specific numbers mentioned in Axioms 2 through 15. These are 0 and 1. The number -1 is implicitly mentioned in Axiom 5. Therefore the following two functions are of interest.

Definition 2.5.1. We define the following two functions:

$$\operatorname{us}(x) := \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases} \quad \text{and} \quad \operatorname{sgn}(x) := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

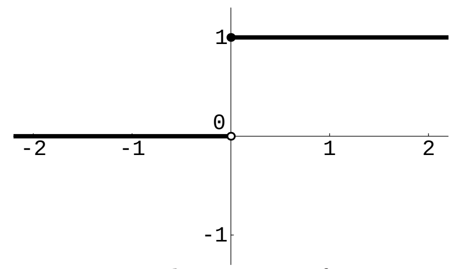


FIGURE 9. The unit step function

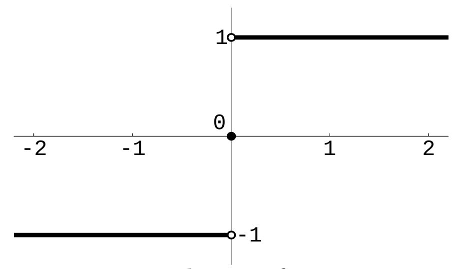


FIGURE 10. The sign function

Definition 2.5.2. The *absolute value* function is defined as

$$\operatorname{abs}(x) = x \operatorname{sgn}(x) \quad \big(\forall x \in \mathbb{R}\big).$$

We will also use the standard notation $\operatorname{abs}(x) = |x|$. The number $|x|$ is called the *absolute value* of the number x.

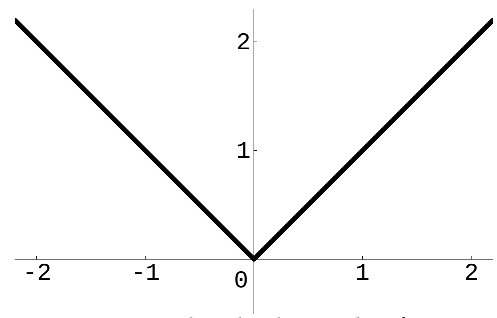


FIGURE 11. The absolute value function

The first function is called the *unit step* (or the *Heaviside step*) function. The second one is called the *sign* function. The definition and the notation for the sign function are standard. However, some authors define the value of the unit step function at 0 to be 1/2. Also, the notation for the unit step function is not standardized; H is often used instead of us. I decided to use two letter notation since it is more in the spirit of sgn and other familiar functions $\sin, \cos, \ln, \exp, \ldots$. Although these two functions are not part of the standard calculus course, I hope that you will agree that they are very simple.

Exercise 2.5.3. Prove the identity: $\operatorname{sgn}(x) = \operatorname{us}(x) - \operatorname{us}(-x)$.

Exercise 2.5.4. Prove the identity: $\operatorname{us}(x) = 1 - \big(\operatorname{sgn}(x) - 1\big)\big(\operatorname{sgn}(x)\big)/2$.

Exercise 2.5.5. Let $x, y \in \mathbb{R}$. Prove the following equalities:

$$\max\{x, y\} = x + (y - x) \operatorname{us}(y - x),$$
$$\min\{x, y\} = y + (x - y) \operatorname{us}(y - x).$$

In the plots above we used a geometric representation of real numbers as points on a straight line. Such representation is obtained by choosing a point on a line to represent 0 and another point to represent 1. Then, every real number corresponds to a point on the line (called the number line), and every point on the number line corresponds to a real number. This geometric representation is often very useful in doing the problems.

Geometrically, the absolute value of a represents the distance between 0 and a, or, generally $|a-b|$ is the *distance* between a and b on the number line.

The basic properties of the absolute value are given in the exercises below. All of the exercises can be proved by considering all possible cases for the numbers involved. This is not difficult when an exercise involves only one number. It gets harder when an exercise involves two or more numbers. Proofs that avoid cases are more elegant and easier to comprehend. Therefore you should always seek such proofs; see Exercise 2.5.9.

Exercise 2.5.6. Prove the following identities.

(a) $|x| = \max\{x, -x\} \quad (\forall\, x \in \mathbb{R})$.
(b) $|x| = x\,\bigl(2\,\mathrm{us}(x) - 1\bigr) \quad (\forall\, x \in \mathbb{R})$.

Exercise 2.5.7. Prove the following statements.

(i) $|a| \geq 0$ for all $a \in \mathbb{R}$.
(ii) $|-a| = |a|$ for all $a \in \mathbb{R}$.
(iii) $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$.

Exercise 2.5.8. Let $x, a \in \mathbb{R}$ and $a \geq 0$. Prove the following equivalences.

(a) $|x| \leq a$ if and only if $-a \leq x$ and $x \leq a$.
(b) $|x| \geq a$ if and only if $x \leq -a$ or $x \geq a$.

Exercise 2.5.9. For all $a, b \in \mathbb{R}$ we have

$$|a+b| \leq |a| + |b|.$$

Solution. By Exercise 2.5.6 (a), $a \leq |a|$ and $b \leq |b|$. Therefore, $a + b \leq |a| + |b|$. Similarly, $-a \leq |a|$ and $-b \leq |b|$. Therefore, $-a - b \leq |a| + |b|$. Since $-a-b = -(a+b)$, we have $-(a+b) \leq |a|+|b|$. Hence, we proved both $a+b \leq |a|+|b|$ and $-(a+b) \leq |a| + |b|$. Therefore,

$$\max\{a+b, -(a+b)\} \leq |a| + |b|. \qquad \square$$

Exercise 2.5.10. Find specific $a, b \in \mathbb{R}$ such that $|a+b| = |a|+|b|$. Next, formulate a general statement by completing the following equivalence

$$|a+b| = |a| + |b| \quad \text{if and only if} \quad \boxed{}\,.$$

Prove your statement.

Exercise 2.5.11. Formulate a general statement by completing the following equivalence

$$|a+b| < |a| + |b| \quad \text{if and only if} \quad \boxed{}\,.$$

Prove your statement.

Exercise 2.5.12. Let $x, y, z \in \mathbb{R}$. Interpret the numbers $|x-y|$, $|y-z|$ and $|x-z|$ as distances and discover an inequality that they must satisfy. (It might help to think of x, y and z as towns on I-5.) Prove your inequality.

Exercise 2.5.13. For all $a, b \in \mathbb{R}$ we have

$$\bigl||a| - |b|\bigr| \leq |a-b|.$$

The inequalities in Exercises 2.5.9, 2.5.12 and 2.5.13 are often called with one name, the *triangle inequality*.

Exercise 2.5.14. Let $x, a \in \mathbb{R}$. If $|x - a| \le 1$, then $|x| \le 1 + |a|$.

Exercise 2.5.15. Let $x, a \in \mathbb{R}$. If $|x - a| \le 1$, then $|x + a| \le 1 + 2|a|$.

Exercise 2.5.16. Let $x, a, u \in \mathbb{R}$ and let $u > 0$. If $|x - a| < u$ and $|x - a| \le 1$, then $\left|x^2 - a^2\right| < u(1 + 2|a|)$.

Exercise 2.5.17. Let $x, a \in \mathbb{R}$ and let $a \neq 0$. If $|x - a| < \dfrac{|a|}{2}$, then $|x| > \dfrac{|a|}{2}$.

Exercise 2.5.18. Let $a \in \mathbb{R}$ and $\epsilon > 0$. Then

$$\bigl\{x \in \mathbb{R} : |x - a| < \epsilon\bigr\} = (a - \epsilon, a + \epsilon).$$

2.6. The set ℕ

We mentioned natural numbers and integers informally in the course of our discussion of the fundamental properties of $\mathbb{R}$. Notice again that the only numbers that are specifically mentioned in Axioms 1 through 15 are 0 and 1. But, in Section 2.2 Exercise 2.2.4 we proved that there are other numbers in $\mathbb{R}$, and we defined the numbers $2, 3, 4, 5, 6, 7, 8$ and 9. The reason that we stopped at 9 is the fact that the number $9 + 1$ plays a special role in our culture. We could continue this process further, but it would not lead to a rigorous definition of the set of natural numbers. Therefore we chose a different route.

Consider the following two properties of a subset S of $\mathbb{R}$:

$$1 \in S, \tag{2.6.1}$$

$$n \in S \quad \Rightarrow \quad n + 1 \in S. \tag{2.6.2}$$

There are many subsets of $\mathbb{R}$ that have these two properties. For example, one such set is the set of positive real numbers, that is the open infinite interval,

$$(0, +\infty).$$

Another such set is the closed infinite interval

$$[1, +\infty),$$

and also the union

$$\{1\} \cup [2, +\infty).$$

There are many such sets. Next we form the family of <u>all</u> subsets of $\mathbb{R}$ with the properties (2.6.1) and (2.6.2):

$$\mathcal{N} := \Bigl\{S \subseteq \mathbb{R} \,:\, 1 \in S \ \text{ and } \ n \in S \ \Rightarrow \ n + 1 \in S\Bigr\}$$

Intuitively, the set of natural numbers is the smallest set in $\mathcal{N}$.

Definition 2.6.1. We define $\mathbb{N}$ to be the intersection of the family $\mathcal{N}$:

$$\mathbb{N} := \bigcap \bigl\{S \,:\, S \in \mathcal{N}\bigr\}.$$

That is, $k \in \mathbb{N}$ if and only if $k \in S$ for all $S \in \mathcal{N}$. The elements of the set $\mathbb{N}$ are called *natural numbers*.

With this definition and Axioms 1 through 15 we should be able to prove all familiar properties of natural numbers.

Theorem 2.6.2. (N 1) $1 \in \mathbb{N}$.
(N 2) *The formula* $\sigma(n) = n + 1$ *defines a function* $\sigma : \mathbb{N} \to \mathbb{N}$.
(N 3) *If* $\sigma(m) = \sigma(n)$, *then* $n = m$; *that is* σ, *is one-to-one.*
(N 4) *For all* $n \in \mathbb{N}$, $\sigma(n) \neq 1$.
(N 5) *If* $K \subseteq \mathbb{N}$ *has the following two properties*

$$1 \in K,$$
$$(\forall n \in \mathbb{N}) \quad n \in K \quad \Rightarrow \quad n + 1 \in K,$$

then $K = \mathbb{N}$.

Proof. Since $1 \in S$ for all $S \in \mathcal{N}$, we have $1 \in \mathbb{N}$. This proves (N 1). To prove (N 2), let $n \in \mathbb{N}$ be arbitrary. Then $n \in S$ for all $S \in \mathcal{N}$. Since (2.6.2) holds for each $S \in \mathcal{N}$, we conclude that $n + 1 \in S$ for all $S \in \mathcal{N}$. Hence $n + 1 \in \mathbb{N}$ for all $n \in \mathbb{N}$. Property (N 3) follows from Exercise 2.1.2 (a). To prove (N 5) assume that $K \subseteq \mathbb{N}$ and K has properties (2.6.1) and (2.6.2). Then $K \in \mathcal{N}$. Consequently, $\mathbb{N} = \bigcap\{S : S \in \mathcal{N}\} \subseteq K$. Thus, $K = \mathbb{N}$. □

Remark 2.6.3. The five properties of $\mathbb{N}$ proved in Theorem 2.6.2 are known as Peano's axioms. Italian mathematician Giuseppe Peano (1858-1932) used these five properties for an axiomatic foundation of natural numbers. All other familiar properties of the natural numbers can be proved using these axioms. The theory of natural numbers developed from Peano's axioms is called Peano's arithmetic.

An important consequence of the property (N 5) in Theorem 2.6.2 is the *Principle of Mathematical Induction*. It is stated and proved in the next theorem. This principle is the main tool in dealing with statements involving natural numbers.

Theorem 2.6.4. *Let* $P(n)$, $n \in \mathbb{N}$, *be a family of statements such that*

(I) $P(1)$ *is true,*
(II) *For all* $n \in \mathbb{N}$, $P(n)$ *implies* $P(n + 1)$.

Then the statement $P(n)$ *is true for each* $n \in \mathbb{N}$.

Proof. Consider the set

$$S = \{n \in \mathbb{N} : P(n) \text{ is true } \}.$$

By (I), $1 \in S$. By (II), for all $n \in \mathbb{N}$, if $n \in S$, then $n + 1 \in S$. Hence, S has both properties from Theorem 2.6.2 (5). Consequently, $S = \mathbb{N}$. This means that for all $n \in \mathbb{N}$ the statement $P(n)$ is true. □

Remark 2.6.5. The step (II) of the mathematical induction requires you to reach the conclusion that $P(n+1)$ is true by using the assumption that $P(n)$ is true, i.e., you have to prove the implication $P(n) \Rightarrow P(n + 1)$ for all $n \in \mathbb{N}$.

The following theorem can be proved using the properties from Theorem 2.6.2 and the principle of mathematical induction.

Theorem 2.6.6. (i) $1 = \min \mathbb{N}$; *that is,* $1 \in \mathbb{N}$ *and* $1 \leq n$ *for all* $n \in \mathbb{N}$.
(ii) *For every* $n \in \mathbb{N}\setminus\{1\}$, *we have* $n - 1 \in \mathbb{N}$.
(iii) *For all* $m, n \in \mathbb{N}$, *we have* $m + n \in \mathbb{N}$.
(iv) *For all* $m, n \in \mathbb{N}$ *we have* $mn \in \mathbb{N}$.
(v) *For all* $m, n \in \mathbb{N}$ *such that* $m < n$, *we have* $n - m \in \mathbb{N}$.
(vi) *If* $m, n \in \mathbb{N}$ *and* $m < n$, *then* $m + 1 \leq n$.

Proof. (i) As we mentioned before the closed infinite interval $[1, +\infty)$ belongs to the family $\mathcal{N}$. Therefore $\mathbb{N} \subseteq [1, +\infty)$. Therefore $n \geq 1$ for all $n \in \mathbb{N}$. Since $1 \in \mathbb{N}$ was proved in Theorem 2.6.2, (i) is proved.

(ii) Consider the following set $S = \{1\} \cup \bigl\{m \in \mathbb{N} : m - 1 \in \mathbb{N}\bigr\}$. Clearly $S \subseteq \mathbb{N}$ and $1 \in S$. Notice also that $2 \in S$, since $2 - 1 = 1 \in \mathbb{N}$. Next we will prove

$$n \in S \Rightarrow n + 1 \in S. \tag{2.6.3}$$

Assume $n \in S$. We distinguish two cases: $n = 1$ and $n \in \bigl\{m \in \mathbb{N} : m - 1 \in \mathbb{N}\bigr\}$. If $n = 1$, then $n + 1 = 2 \in S$. Hence (2.6.3) holds in this case. If $n \in \bigl\{m \in \mathbb{N} : m - 1 \in \mathbb{N}\bigr\}$, then $n \in \mathbb{N}$ and $n - 1 \in \mathbb{N}$. By Theorem 2.6.2 (N 2), $n + 1 \in \mathbb{N}$ and, obviously, $(n + 1) - 1 = n \in \mathbb{N}$. Therefore $n + 1 \in \bigl\{m \in \mathbb{N} : m - 1 \in \mathbb{N}\bigr\}$. Hence $n + 1 \in S$. Thus (2.6.3) holds. Now, by Theorem 2.6.2 (5), $S = \mathbb{N}$. This proves $\mathbb{N}\backslash\{1\} = \bigl\{m \in \mathbb{N} : m - 1 \in \mathbb{N}\bigr\}$.

Remaining properties are proved similarly. □

The Principle of Mathematical Induction is also used to define functions on $\mathbb{N}$. The process described in the next proposition is called the *Principle of Inductive Definition.*

Proposition 2.6.7. *If a function f has the following two properties*

(I) $f(1)$ *is defined,*
(II) $\bigl(\forall\, n \in \mathbb{N}\bigr) \quad f(n+1) \quad$ *is defined in terms of* $\quad f(1), \ldots, f(n)$,

then f is defined on $\mathbb{N}$.

Proof. Denote the domain of f by D. Let $k \in \mathbb{N}$ and set the statement $P(k)$ to be: $1, \ldots, k \in D$. Clearly $P(1)$ is true by (I). Now, let $n \in \mathbb{N}$ be arbitrary and assume that $P(n)$ is true. That is assume that $1, \ldots, n \in D$. By (II) $f(n+1)$ is defined in terms of $f(1), \ldots, f(n)$. Since by the inductive hypothesis all $f(1), \ldots, f(n)$ are defined, we conclude that $f(n+1)$ is defined. Thus $n+1 \in D$. Since we assume that $1, \ldots, n \in D$, we have proved that $1, \ldots, n, n+1 \in D$. Hence $P(n+1)$ is proved. By the Principle of Mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$. Therefore $1, \ldots, n \in D$ for all $n \in \mathbb{N}$. Consequently, $n \in D$ for all $n \in \mathbb{N}$. □

A definition of a function with properties (I) and (II) in Proposition 2.6.7 is called *recursive* or *inductive* definition.

Definition 2.6.8. A function whose domain equals $\mathbb{N}$ and whose range is in $\mathbb{R}$ is called a *sequence in* $\mathbb{R}$.

Remark 2.6.9. Traditionally, if $f : A \to B$ is a function and if $x \in A$, then the value of f at x is denoted by $f(x)$. In addition to this traditional notation, for a sequence $f : \mathbb{N} \to \mathbb{R}$ we will often write f_n instead of $f(n)$, $n \in \mathbb{N}$. When convenient we will use both notations for the same sequence. The reason for this is purely typographical. For example if $n = \dfrac{m(m+1)}{2} + 1$, then it is awkward to write $f_{\frac{m(m+1)}{2}+1}$. In such a case, the expression $f\bigl(\frac{m(m+1)}{2} + 1\bigr)$ is preferable since it is easier to read and understand.

2.7. Examples and Exercises related to $\mathbb{N}$

The following two examples deal with two familiar functions: the factorial and the power function. Let $n \in \mathbb{N}$. The factorial is informally "defined" as

$$n! = 1 \cdot 2 \cdot \ldots \cdot (n-1) \cdot n.$$

Let $a \in \mathbb{R}$. The n-th power of a is informally expressed as

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a \cdot a}_{n \text{ times}}.$$

Next we give the rigorous definitions of the factorial and the power function as examples of recursive definitions.

Example 2.7.1. The function $f : \mathbb{N} \to \mathbb{N}$ defined by

(i) $f(1) = 1$,
(ii) $\bigl(\forall\, n \in \mathbb{N}\bigr)\;\; f(n+1) = (n+1)\, f(n)$,

is called the *factorial*.

The standard notation for the factorial is $f(n) = n!$. The definition of factorial is extended to 0 by setting $0! = 1$.

Example 2.7.2. Let $a \in \mathbb{R}$. Define the function $g : \mathbb{N} \to \mathbb{R}$ by

(i) $g(1) = a$,
(ii) $\bigl(\forall\, n \in \mathbb{N}\bigr)\;\; g(n+1) = a\, g(n)$.

The standard notation for the function g is $g(n) = a^n$. The expression a^n is called the n-th *power* of a. For $a \neq 0$, the definition of the power is extended to 0 by setting $a^0 = 1$. The expression 0^0 is not defined.

Exercise 2.7.3. Let $a, b \in \mathbb{R}$ be such that $a, b \geq 0$. Let $n \in \mathbb{N}$. Prove that $a < b$ if and only if $a^n < b^n$.

Use the Principle of Mathematical Induction to do the following exercises.

Exercise 2.7.4. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by

(i) $f(1) = 1$,
(ii) $\bigl(\forall\, n \in \mathbb{N}\bigr)\;\; f(n+1) = f(n) + (2n+1)$.

Evaluate the values $f(2), f(3), f(4), f(5)$. Based on the numbers that you get, guess a simple formula for $f(n)$ and prove it.

Exercise 2.7.5. Consider the function $T : \mathbb{N} \to \mathbb{N}$ defined by

(i) $T(1) = 1$,
(ii) $\bigl(\forall\, n \in \mathbb{N}\bigr)\;\; T(n+1) = T(n) + (n+1)$.

Evaluate the values $T(2), T(3), T(4), T(5), T(6)$. Based on these numbers guess a simple formula for $T(n)$ in terms of n and prove it.

Remark 2.7.6. The numbers $T(n)$, $n \in \mathbb{N}$, are called *triangular numbers*. For $n \in \mathbb{N}$, the triangular number

$$T(n) = 1 + 2 + \cdots + (n-1) + n$$

is the additive analog of the factorial (see Example 2.7.1)

$$n! = 1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n.$$

For completeness we set $T(0) = 0$.

Exercise 2.7.7. Let $a, x \in \mathbb{R}$. Consider the function $g : \mathbb{N} \to \mathbb{R}$ defined by

(i) $g(1) = a$,
(ii) $\bigl(\forall\, n \in \mathbb{N}\bigr)\quad g(n+1) = g(n) + a\,x^n$.

Another way of writing $g(n)$ is

$$g(n) = \sum_{k=0}^{n-1} a\,x^k.$$

Informally this sum is sometimes written as

$$g(n) = a + a\,x + \cdots + a\,x^{n-1}.$$

This sum is called the *geometric sum*.

Prove that

$$g(n) = \begin{cases} a\,\dfrac{1-x^n}{1-x} & \text{if } \ x \neq 1, \\[2ex] n\,a & \text{if } \ x = 1. \end{cases}$$

Exercise 2.7.8 (Bernoulli's inequality)**.** Let $n \in \mathbb{N}$ and $x > -1$. Then

$$(1+x)^n \geq 1 + n\,x.$$

Exercise 2.7.9. Let $n \in \mathbb{N}$ and let $x \in \mathbb{R}$ be such that $0 \leq x \leq 1$. Then

$$(1+x)^n \leq 1 + \bigl(2^n - 1\bigr)x.$$

Exercise 2.7.10 (Binomial theorem)**.** Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}\, y^k.$$

Here $\binom{n}{k}$ denotes the *binomial coefficient* which is defined by

$$\binom{n}{k} := \frac{n!}{k!\,(n-k)!}, \qquad n \in \mathbb{N}, \quad k = 0, 1, \ldots, n.$$

The most important property of binomial coefficients is given by the following equality

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}, \qquad n \in \mathbb{N}, \quad k = 1, \ldots, n.$$

This formula is proved by using the definition of the binomial coefficients and the rules for adding fractions.

2.8. Finite sets, infinite sets, countable sets

One of the most important applications of the natural numbers is counting. The following special subsets of $\mathbb{N}$ are used for counting

$$[\![1, n]\!]_{\mathbb{N}} := \bigl\{k \in \mathbb{N} \,:\, k \leq n\bigr\}, \quad n \in \mathbb{N}.$$

Since this notation will be used often in this section, we will, for simplicity of notation drop the subscript $\mathbb{N}$ in $[\![1,n]\!]_{\mathbb{N}}$ and simply write $[\![1,n]\!]$. For example

$$\begin{aligned}
[\![1,1]\!] &= \{1\} \\
[\![1,2]\!] &= \{1,2\} \\
[\![1,3]\!] &= \{1,2,3\} \\
[\![1,4]\!] &= \{1,2,3,4\} \\
[\![1,5]\!] &= \{1,2,3,4,5\} \\
[\![1,6]\!] &= \{1,2,3,4,5,6\} \\
&\vdots
\end{aligned}$$

Next we give a formal mathematical definition of the counting process.

Definition 2.8.1. A set A is *finite* if there exists a natural number n and a bijection $f : [\![1,n]\!] \to A$. In this case we say that A *has* n *elements*. We use the notation $\#A$ for the number of elements of A.

Exercise 2.8.2. If A is finite and $b \notin A$, then $A \cup \{b\}$ is finite and and $\#\big(A \cup \{b\}\big) = \#A + 1$.

Exercise 2.8.3. A nonempty set A is finite if and only if there exists $n \in \mathbb{N}$ and a surjection $f : [\![1,n]\!] \to A$. In this case $\#A \leq n$. Hint: One direction here is trivial. The other direction is a statement involving a natural number, so it can be proved by mathematical induction. True for $n = 1$. Assume true for n and $f : [\![1,n+1]\!] \to A$ surjection. Set $B = f\big([\![1,n]\!]\big)$. Then by the inductive hypothesis B is finite and $\#B \leq n$. Two cases $B = A$. Then, A finite and $\#A \leq n \leq n+1$. If $B \subset A$. Then $A = B \cup \{f(n+1)\}$. By Exercise 2.8.2 A is finite and $\#A = 1 + \#B \leq n+1$.

Finite sets are often informally written as $A = \{a_1, a_2, \ldots, a_n\}$. However, this way of writing does not imply that the mapping $k \mapsto a_k$, $k \in [\![1,n]\!]$, is a bijection, but it does imply that this mapping is a surjection.

Exercise 2.8.4. A nonempty subset B of a finite set A is finite. We have $\#B \leq \#A$. Hint: Let $f : [\![1,n]\!] \to A$ be a bijection. Since B is not empty there exists $b \in B$. Define the function $g : A \to B$ by $g(x) = \begin{cases} x & \text{if } x \in B \\ b & \text{if } x \in A \setminus B \end{cases}$. Clearly the range of g is B. Therefore $g \circ f : [\![1,n]\!] \to B$ is a surjection. By Exercise 2.8.3 B is finite.

Exercise 2.8.5. Let B is a nonempty proper subset of a finite set A. Then $\#B < \#A$. Hint: Let $c \in A \setminus B$. Then

$$\#B \leq \#\big(A \setminus \{c\}\big) = \#A - 1 < \#A.$$

Exercise 2.8.6. If A is a finite subset of $\mathbb{R}$, then A has a minimum and a maximum.

Definition 2.8.7. A nonempty set which is not finite is said to be *infinite.*

Remark 2.8.8. The previous definition of an infinite set is certainly logically correct, but it is not "constructive".

Therefore it is desirable to give a formal negation of the definition of finite set. Before doing that I will restate the definition of a finite set as:

"A set A is finite if (and only if) there exists $n \in \mathbb{N}$ and there exists an bijection $f : [\![1, n]\!] \to A$."

The negation of the last statement (and thus a characterization of an infinite set) is the following

"For every $n \in \mathbb{N}$ and for every $f : [\![1, n]\!] \to A$ we have that f is not a bijection."

Remark 2.8.9. The importance of Exercise 2.8.6 is twofold. First, it states the most important property of finite sets of real numbers. Second, its contrapositive provides a simple way of proving that a set is infinite: If a nonempty subset of $\mathbb{R}$ does not have a minimum or it does not have a maximum, then it is infinite.

The fact that infinite sets might not have a minimum and/or maximum makes dealing with such sets more difficult. The following proposition states that not having a minimum and/or maximum is to some extend a universal property of an infinite subset of $\mathbb{R}$.

Proposition 2.8.10. *Let $A \subset \mathbb{R}$. If A is infinite, then there exists a nonempty subset B of A such that B does not have a minimum or there exists a nonempty subset C of A such that C does not have a maximum.*

Proof. We will prove the equivalent implication: If A is an infinite subset of $\mathbb{R}$ and each nonempty subset of A has a minimum, then there exist a nonempty subset C of A such that C does not have a maximum.

So, assume that A is an infinite subset of $\mathbb{R}$ and each nonempty subset of A has a minimum. Then, in particular, $\min A$ exists. Let W be the set of all minimums of infinite subsets of A. Formally,

$$W = \Bigl\{ x \in A \ : \ x = \min E \ \text{ where } \ E \subset A \ \text{ and } \ E \text{ is infinite} \Bigr\}.$$

Clearly $\min A$ is an element in W. Hence $W \neq \emptyset$.

Next we will prove that W does not have a maximum. Let $y \in W$ be arbitrary. Then there exists an infinite subset F of A such that $y = \min F$. Since F is infinite, the set $F \setminus \{y\}$ is also infinite. Since $F \setminus \{y\} \subset A$, by the assumption $z = \min\bigl(F \setminus \{y\}\bigr)$ exists. Therefore, $z \in W$. Since $z \in F \setminus \{y\}$, we have $z \neq y$. Since $z \in F$ and $y = \min F$, we have $z \geq y$. Hence $z > y$. Thus, for each $y \in W$ there exists $z \in W$ such that $z > y$. This proves that W is a nonempty subset of A which does not have a maximum. □

Remark 2.8.11. The above proposition is an implication of the form: $P \Rightarrow Q \vee R$. This implication is equivalent to the implication $P \wedge \neg Q \Rightarrow R$. One way to see this is to consider the negations of both implications. The negation of $P \Rightarrow Q \vee R$ is $P \wedge (\neg Q \wedge \neg R)$, while the negation of $P \wedge \neg Q \Rightarrow R$ is $(P \wedge \neg Q) \wedge \neg R$. Since the negations are clearly equivalent, the implications are also equivalent.

Exercise 2.8.12. Prove that $\mathbb{N}$ does not have a maximum.

Exercise 2.8.13. Prove that a nonempty subset of $\mathbb{N}$ is finite if and only if it has a maximum.

Exercise 2.8.14. Prove that the set $\mathbb{N}$ is infinite.

Exercise 2.8.15. Let A be a nonempty subset of $\mathbb{N}$. Then A has a minimum.

Remark 2.8.16. Subsets of $\mathbb{N}$ can be infinite. As I mentioned in Remark 2.8.9 a problem with infinite sets is a possible absence of minimum and maximum. Exercise 2.8.15 tells us that a subset of natural numbers must at least have a minimum. Consequently, infinite subsets of $\mathbb{N}$ are not as bad as infinite subsets of $\mathbb{R}$.

Solution of Exercise 2.8.15. This proof uses the following two facts:

(1) Each finite set has a minimum. (Proved in Exercise 2.8.6.)
(2) For each $n \in \mathbb{N}$ each subset of the set $\{1, 2, \ldots, n\} = [\![1, n]\!]$ is finite. (Proved in Exercise 2.8.4.)

Since $A \neq \emptyset$, there exists $n \in A$. Consider the set $B = \{x \in A : x \leq n\}$. Then $B \subseteq [\![1, n]\!]$. By fact (2) B is finite. Now, by fact (1) B has a minimum; denote it by $m = \min B$. Then m is also the minimum of A. (Here is a proof: If $a \in A$, then either $a \leq n$, or $n < a$. In the first case $a \in B$, and therefore $m \leq a$. If $n < a$, then $m \leq n < a$, and therefore $m \leq a$ for each $a \in A$.) □

Definition 2.8.17. A set A is *countable* if there exists a bijection $f : \mathbb{N} \to A$.

Exercise 2.8.18. Prove that the set of even natural numbers is countable.

Exercise 2.8.19. If S is an infinite subset of $\mathbb{N}$, then S is countable.

Solution. (This is an extended Hint.) Let S be an infinite subset of $\mathbb{N}$. Let $s \in S$ be arbitrary. Then the set $S \cap [\![1, s]\!]$ is finite, since it is a nonempty subset of the finite set $[\![1, s]\!]$. Define the function:

$$f(s) := \#\bigl(S \cap [\![1, s]\!]\bigr), \quad s \in S.$$

Clearly $f : S \to \mathbb{N}$. The function f has the following three properties:

(I) If $s, t \in S$ and $s < t$, then $f(s) < f(t)$.
(II) If $s = \min S$, then $f(s) = 1$.
(III) If $s \in S$ and $t = \min\bigl(S \setminus [\![1, s]\!]\bigr)$, then $f(t) = f(s) + 1$.

Property (I) follows from Exercise 2.8.5. Property (II) follows from the fact that, $s = \min S$ implies $S \cap [\![1, s]\!] = \{s\}$. Property (III) follows from Exercise 2.8.2.

Property (I) implies that f is an injection. Properties (II) and (III) imply that the range of f, call it T, has the following properties: $1 \in T$ and $n \in T \Rightarrow n+1 \in T$. Since $T \subseteq \mathbb{N}$, this, by Theorem 2.6.2 (5), implies $T = \mathbb{N}$. Thus f is a surjection. Hence, f is a bijection. □

It will be proved in Section 2.10 that the set of integers and the set of rational numbers are countable sets.

A difficulty in proving that a particular set is countable is in the fact that we have to construct a bijection between $\mathbb{N}$ and that set. It turns out that a surjection suffices. That is the content of the next proposition. It states that it is sufficient to construct a surjection of $\mathbb{N}$ onto that set. This will be used to prove that the set of rational numbers is countable.

Proposition 2.8.20. *Let A be an infinite set and let $g : \mathbb{N} \to A$ be a surjection. Then A is countable. That is, there exists a bijection $\phi : \mathbb{N} \to A$.*

Proof. Assume that A is an infinite set and that $g : \mathbb{N} \to A$ is a surjection. Since g is a surjection the set $\bigl\{k \in \mathbb{N} : g(k) = a\bigr\}$ is nonempty for each $a \in A$. By Exercise 2.8.15 this set has a minimum. Define the function

$$h(a) := \min\bigl\{k \in \mathbb{N} : g(k) = a\bigr\}, \quad a \in A.$$

Clearly $h : A \to \mathbb{N}$. As an exercise the reader can prove that h is one-to-one. Denote by $S \subseteq \mathbb{N}$ the range of h. Then h is a bijection between A and S. Since A is infinite, S is also infinite. (Prove this as an exercise.) Therefore, by Exercise 2.8.19, there exists a bijection $f : S \to \mathbb{N}$. Now, $h : A \to S$ is a bijection and $f : S \to \mathbb{N}$. Hence the composition $f \circ h$ is also a bijection. Since $f \circ h : A \to \mathbb{N}$ the proof is complete. □

We conclude with a simpler exercise in the same spirit.

Exercise 2.8.21. If A is an infinite set and there is an injection $f : A \to \mathbb{N}$, then A is countable.

2.9. More on countable sets

In Exercise 2.7.5 we defined the sequence of triangular numbers: $T_1 = 1, T_2 = 3, T_3 = 6, T_4 = 10, \ldots$. In the following example we give a definition of the sequence indicated by the following table. The **triangular numbers** are in bold face.

	T_1		T_2			T_3				T_4					T_5						T_6	
n	**1**	2	**3**	4	5	**6**	7	8	9	**10**	11	12	13	14	**15**	16	17	18	19	20	**21**	22
R_n	1	2	2	3	3	3	4	4	4	4	5	5	5	5	5	6	6	6	6	6	6	7

Example 2.9.1. Define the sequence $R : \mathbb{N} \to \mathbb{N}$ by

$$R_n = \min\bigl\{k \in \mathbb{N} : T_k \geq n\bigr\}, \qquad n \in \mathbb{N}.$$

Remark 2.9.2. Clearly $n(n+1) \geq 2n$ for all $n \in \mathbb{N}$. Therefore, $T_n \geq n$ for all $n \in \mathbb{N}$. Consequently the set in the definition of R_n is not empty. By Exercise 2.8.15 every nonempty subset of $\mathbb{N}$ has a minimum, so R_n is well defined.

Next we will prove that the sequence $\{R_n\}$ really does what is indicated in the table at the beginning of this section.

Exercise 2.9.3. Let $n, m \in \mathbb{N}$. Then $R_n = m$ if and only if

$$\frac{(m-1)m}{2} + 1 \leq n \leq \frac{m(m+1)}{2}. \tag{2.9.1}$$

Solution. Assume that $R_n = m$. Then $m = \min\bigl\{k \in \mathbb{N} : T_k \geq n\bigr\}$. In particular $T_m \geq n$ and $T_{m-1} < n$. This proves (2.9.1). Now assume (2.9.1). If $m = 1$, then $n = 1$ and $R_1 = 1$ is true. Next assume $m > 1$. Then, $n \leq T_m$, so $R_n \leq m$. Since it is easy to prove that $T_j \leq T_{m-1}$ for all $j \in \{1, \ldots, m-1\}$, we have $T_k < n$ for all $k \in \{1, \ldots, m-1\}$. Hence $R_n \geq m$. □

Remark 2.9.4. There are several other formulas for the sequence R. For example for $n \in \mathbb{N}$,

$$R_n = \left\lfloor \frac{1}{2} + \sqrt{2n} \right\rfloor \quad \text{or} \quad R_n = \left\lceil -\frac{1}{2} + \sqrt{2n} \right\rceil.$$

Here $\lfloor \cdot \rfloor$ is the floor function, $\lceil \cdot \rceil$ is the ceiling function and $\sqrt{\cdot}$ is the square root function. These functions will be introduced in Sections 2.12 and 2.13.

An alternative way to define the sequence R is the following recursive definition.

(i) $R_1 = 1$,

(ii) $\bigl(\forall\, n \in \mathbb{N}\bigr) \quad R_{n+1} = 1 + R(n + 1 - R_n)$.

The following exercise deals with the Cartesian square of the set $\mathbb{N}$; that is the set $\mathbb{N}\times\mathbb{N}$. Recall that this is the set of all ordered pairs of positive integers:

$$\mathbb{N}\times\mathbb{N} := \bigl\{(s,t)\,:\, s,t\in\mathbb{N}\bigr\}.$$

The set $\mathbb{N}\times\mathbb{N}$ is illustrated by the following infinite table:

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \dots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \dots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \dots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \dots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Exercise 2.9.5. Prove that the function $A:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ defined by

$$A(s,t) = \frac{(s+t-2)(s+t-1)}{2} + s, \quad s,t\in\mathbb{N},$$

is a bijection.

A LONG HINT: Prove that the inverse of A is given by

$$B(n) = \left(n - \frac{(R_n-1)R_n}{2}, \frac{R_n(R_n+1)}{2} - n + 1\right), \quad n\in\mathbb{N}.$$

Here R is the sequence recursively defined in Example 2.9.1. Notice also that by Exercise 2.7.5 the formulas for A and B can be written as

$$\begin{aligned}
A(s,t) &= T(s+t-2)+s, \quad s,t\in\mathbb{N},\\
B(n) &= \Bigl(n - T\bigl(R_n-1\bigr), T\bigl(R_n\bigr) - n + 1\Bigr), \quad n\in\mathbb{N}.
\end{aligned}$$

Let $s,t\in\mathbb{N}$. To evaluate $B\bigl(A(s,t)\bigr)$ you will need to evaluate $R\bigl(T(s+t-2)+s\bigr)$ first. For this, use Exercise 2.9.3 and the following inequalities

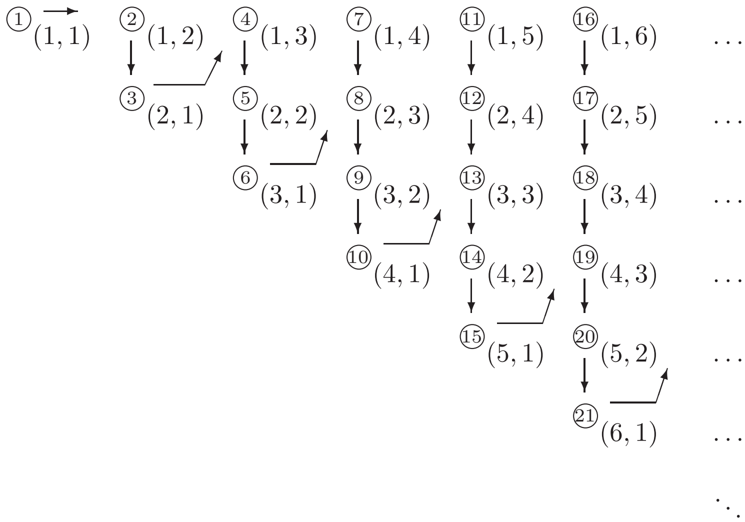
$$T(s+t-2)+1 \le T(s+t-2)+s \le T(s+t-2)+s+t-1 = T(s+t-1),$$

to conclude that

$$R\bigl(T(s+t-2)+s\bigr) = s+t-1.$$

Hence, $R\bigl(A(s,t)\bigr) = s+t-1$. With this identity calculating $B\bigl(A(s,t)\bigr)$ should be easier. This is the end of HINT.

To visualize the action of the function A on $\mathbb{N}\times\mathbb{N}$ we rearrange the table preceding Exercise 2.9.5 in a triangular shape and place the value of A in a circle next to the corresponding ordered pair. As a result we get the following table.



Exercise 2.9.6. Let $\mathcal{A}$ be countable family of sets. Assume that each set in $\mathcal{A}$ is countable. Prove that $\bigcup\{A : A \in \mathcal{A}\}$ is also countable.

Exercise 2.9.7. Prove that the set of all functions $f : \mathbb{N} \to \{0,1\}$ is not countable. (As mentioned in Section 1.6 this set of functions is denoted by $\{0,1\}^{\mathbb{N}}$.)

Solution. To prove the claim we will take an arbitrary function $\Phi : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$ and prove that Φ is not a surjection. This will be proved by constructing a specific element $f \in \{0,1\}^{\mathbb{N}}$, that is $f : \mathbb{N} \to \{0,1\}$, such that $f \neq \Phi_n$ for all $n \in \mathbb{N}$. To construct f let us analyze Φ_n, $n \in \mathbb{N}$. Clearly $\Phi_1 : \mathbb{N} \to \{0,1\}$, that is $\Phi_1(n) \in \{0,1\}$ for all $n \in \mathbb{N}$. We can indicate the action of Φ_1 on $\mathbb{N}$ by listing its first 7 values:

$$\Phi_1(1) \quad \Phi_1(2) \quad \Phi_1(3) \quad \Phi_1(4) \quad \Phi_1(5) \quad \Phi_1(6) \quad \Phi_1(7) \quad \ldots$$

We can do the same for $\Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6, \ldots$ to get

$$\begin{array}{cccccccc}
\boxed{\Phi_1(1)} & \Phi_1(2) & \Phi_1(3) & \Phi_1(4) & \Phi_1(5) & \Phi_1(6) & \Phi_1(7) & \ldots \\
\Phi_2(1) & \boxed{\Phi_2(2)} & \Phi_2(3) & \Phi_2(4) & \Phi_2(5) & \Phi_2(6) & \Phi_2(7) & \ldots \\
\Phi_3(1) & \Phi_3(2) & \boxed{\Phi_3(3)} & \Phi_3(4) & \Phi_3(5) & \Phi_3(6) & \Phi_3(7) & \ldots \\
\Phi_4(1) & \Phi_4(2) & \Phi_4(3) & \boxed{\Phi_4(4)} & \Phi_4(5) & \Phi_4(6) & \Phi_4(7) & \ldots \\
\Phi_5(1) & \Phi_5(2) & \Phi_5(3) & \Phi_5(4) & \boxed{\Phi_5(5)} & \Phi_5(6) & \Phi_5(7) & \ldots \\
\Phi_6(1) & \Phi_6(2) & \Phi_6(3) & \Phi_6(4) & \Phi_6(5) & \boxed{\Phi_6(6)} & \Phi_6(7) & \ldots \\
\Phi_7(1) & \Phi_7(2) & \Phi_7(3) & \Phi_7(4) & \Phi_7(5) & \Phi_7(6) & \boxed{\Phi_7(7)} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Now we are ready to define $f : \mathbb{N} \to \{0,1\}$ which will differ from each Φ_n, $n \in \mathbb{N}$. Set

$$f(n) := 1 - \Phi_n(n), \qquad n \in \mathbb{N}.$$

Since $\Phi_n(n) \in \{0,1\}$ we have that $f(n) = 1 - \Phi_n(n) \neq \Phi_n(n)$ for all $n \in \mathbb{N}$. Therefore

$$f \neq \Phi_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence $\Phi : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$ is not a surjection. Since $\Phi : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$ was arbitrary function, we conclude that there does not exist a bijection between $\mathbb{N}$ and $\{0,1\}^{\mathbb{N}}$. □

2.10. The sets $\mathbb{Z}$ and $\mathbb{Q}$

We define an *integer* to be a real number x such that either $x = 0$ or x is a natural number or $-x$ is a natural number. The set of all integers is denoted by $\mathbb{Z}$. Hence

$$\mathbb{Z} = \bigl\{ x \in \mathbb{R} : \quad x \in \mathbb{N} \quad \text{or} \quad x = 0 \quad \text{or} \quad -x \in \mathbb{N} \bigr\}.$$

Exercise 2.10.1. Prove that $\mathbb{Z}$ is countable.

Exercise 2.10.2. Prove that $\mathbb{Z} \times \mathbb{N}$ is countable.

Now we define a *rational number* to be a real number of the form $m \cdot \frac{1}{n}$ where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. (We write shortly m/n or $\frac{m}{n}$ instead of $m \cdot \frac{1}{n}$.) The set of all rational numbers we denote by $\mathbb{Q}$, that is

$$\mathbb{Q} = \left\{ x \in \mathbb{R} \ : \ \exists\, m \in \mathbb{Z} \text{ and } \exists\, n \in \mathbb{N} \ \text{ such that } \ x = \frac{m}{n} \right\}.$$

Exercise 2.10.3. Denote by $\mathbb{Q}_+$ the set of all positive rational numbers. Prove that $\mathbb{Q}_+$ is countable.

Exercise 2.10.4. Prove that there exists a bijection between $\mathbb{Q}_+$ and $\mathbb{Q}$.

Exercise 2.10.5. Prove that the set $\mathbb{Q}$ is countable.

Exercise 2.10.6. Prove that $r^2 \neq 2$ for all $r \in \mathbb{Q}$.

Solution. Now that we have a definition of $\mathbb{Q}$ we can prove that for each $r \in \mathbb{Q}$ there exist $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ which are not both even such that $r = p/q$. Let $r \in \mathbb{Q}$ be arbitrary. Set

$$S = \left\{ n \in \mathbb{N} : \exists\, m \in \mathbb{Z} \ \text{ such that } \ r = \frac{m}{n} \right\}.$$

Since $r \in \mathbb{Q}$ the set S is not empty. Since S is a nonempty subset of $\mathbb{N}$, by Exercise 2.8.15 S has a minimum. Set $q = \min S$. Since $q \in S$, there exists $p \in \mathbb{Z}$ such that $r = p/q$. Next we will prove that p and q are not both even. That is we will prove the following implication: If $q = \min S$, $p \in \mathbb{Z}$, and $r = p/q$, then p and q are not both even.

It is easier to prove a partial contrapositive of the last implication. If $n \in S$, $m \in \mathbb{Z}$, $r = m/n$ and both m and n are even, then n is not a minimum of S. So, assume $n \in S$, $m \in \mathbb{Z}$ are both even and $r = m/n$. Then there exist $k \in \mathbb{Z}$ and $j \in \mathbb{N}$ such that $m = 2k$ and $n = 2j$. Clearly $j < n$. Also,

$$r = \frac{m}{n} = \frac{2\,k}{2\,j} = \frac{k}{j}.$$

Hence $j \in S$ and therefore n is not a minimum of S.

In Chapter 1 we indicated how to prove that $2\,q^2 \neq p^2$. Hence $r^2 \neq 2$ is proved. □

There are many basic properties on $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ that are not stated so far. For example:

- The set $\mathbb{N}$ is not bounded.

- There exists $\alpha \in \mathbb{R}$ such that $\alpha^2 = 2$.
- The set $\mathbb{Q}$ is a proper subset of $\mathbb{R}$.
- The set $\mathbb{R}$ is not countable.

In Exercise 2.10.6 we proved that the fact that there is no rational number x such that $x^2 = 2$. Since the set $\mathbb{Q}$ of rational numbers satisfies all Axioms 1 through 15 we cannot expect that based only on Axioms 1 through 15 we can prove that there exists a real number number α such that $\alpha^2 = 2$. Therefore we need an extra axiom for the set of real numbers; an axiom that will not be satisfied by the set of rational numbers. This is the **Completeness Axiom** which we introduce in Section 2.11. It turns out that proofs of the four statements listed above use the Completeness Axiom.

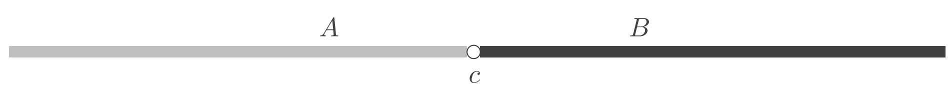
2.11. The Completeness axiom

Recall Exercise 2.2.5: If a and b are real numbers such that $a < b$, then there exists a real number c such that $a < c < b$.

This statement assures that there are no big holes in $\mathbb{R}$; between any two real numbers there is another real number. A natural question to ask is whether the same is true for sets. If we are given two sets which are in some sense separated, does there exists a real number between them? Somewhat surprisingly this has to be postulated as the last axiom of the real numbers:

Axiom 16 (CA: Completeness Axiom)**.** If A and B are nonempty subsets of $\mathbb{R}$ such that for every $a \in A$ and for every $b \in B$ we have $a \leq b$, then there exists $c \in \mathbb{R}$ such that $a \leq c \leq b$ for all $a \in A$ and all $b \in B$.

Visually this corresponds to the picture



Since we perceive the real number line to have no holes, the place marked by the open circle must correspond to a real the number c.

Now we have 16 axioms of $\mathbb{R}$. It is remarkable that all statements about real numbers that are studied in beginning mathematical analysis courses can be deduced from these sixteen axioms and basic properties of sets.

The formulation of the Completeness Axiom given as Axiom 16 above is not standard. This formulation I found in the book *Mathematical analysis* by Vladimir Zorich, published by Springer in 2004. The standard formulation of the Completeness Axiom is given in Exercise 2.13.5 below. In that exercise you will prove that Zorich's Completeness Axiom is equivalent to the standard one.

Now we have a powerful tool. Let us use it to prove some important statements. We start with the proof that $\mathbb{N}$ is not bounded.

Exercise 2.11.1 (Archimedean Property)**.** For every $b \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $n > b$.

Solution. We will prove the statement by contradiction. Assume that the negation of the statement is true. That is, assume that there exists $\beta \in \mathbb{R}$ such that $\beta \geq n$ for all $n \in \mathbb{N}$. Set

$$A = \mathbb{N} \quad \text{and} \quad B = \big\{ b \in \mathbb{R} : b \geq n \ \ \forall n \in \mathbb{N} \big\}.$$

Since $1 \in \mathbb{N}$ and $\beta \in B$, the sets A and B are nonempty subsets of $\mathbb{R}$. By the definition of the set B we have $a \leq b$ for all $a \in A$ and all $b \in B$. By Completeness Axiom there exists $c \in \mathbb{R}$ such that $a \leq c \leq b$ for all $a \in A$ and all $b \in B$. In other words, $n \leq c \leq b$ for all $n \in \mathbb{N}$ and all $b \in B$. Since $c \leq b$ for all $b \in B$, we conclude that $c - 1/2 \notin B$. Thus, there exists $m \in \mathbb{N}$ such that $c - 1/2 < m$. Since $n \leq c$ for all $n \in \mathbb{N}$ we conclude that $m + 1 \leq c$. Hence,

$$c - 1/2 < m < m + 1 \leq c.$$

Using the above inequalities we get

$$1 = (m+1) - m < c - \big(c - 1/2\big) = 1/2,$$

that is $2 < 1$. Wrong! (by Exercise 2.2.4) This proves the statement. □

Exercise 2.11.2. Prove that there exists a unique positive real number α such that $\alpha^2 = 2$.

Solution. Set

$$A = \big\{ x \in \mathbb{R} : x > 0 \text{ and } x^2 < 2 \big\}, \qquad B = \big\{ y \in \mathbb{R} : y > 0 \text{ and } y^2 > 2 \big\}.$$

Since $1 \in A$ and $2 \in B$, A and B are nonempty subsets of $\mathbb{R}$. By Exercise 2.2.7, $x < y$ for all $x \in A$ and all $y \in B$. The Completeness axiom implies that there exists $c \in \mathbb{R}$ such that $x \leq c \leq y$ for all $x \in A$ and all $y \in B$.

Next we will prove that B does not have minimum and A does not have maximum. The idea for these proofs comes from Exercise 2.2.9 which states that $(s+t)^2 > 4st$ whenever $s \neq t$. Thus for arbitrary $s > 0$ such that $s^2 \neq 2$ we have

$$\left(\frac{s}{2} + \frac{1}{s} \right)^2 > 2. \tag{2.11.1}$$

That is $(s/2 + 1/s) \in B$. Taking reciprocal and multiplying by 4 in (2.11.1) yields

$$\frac{4}{\left(\frac{s}{2} + \frac{1}{s}\right)^2} < 2. \tag{2.11.2}$$

Hence $2/(s/2 + 1/s) \in A$.

A proof that B does not have a minimum follows. Let $y \in B$. Then $y^2 > 2$ and $y > 0$. Set

$$v = \frac{y}{2} + \frac{1}{y}.$$

Clearly $v > 0$. Since $y^2 > 2$ we have $y/2 > 1/y$. Hence $y > y/2 + 1/y$, that is $y > v$. Since by (2.11.1), $v \in B$ this proves that B does not have a minimum.

A proof that A does not have a maximum follows. Let $x \in A$. Then $x^2 < 2$ and $x > 0$. Set

$$u = \frac{2}{\frac{x}{2} + \frac{1}{x}}.$$

Clearly $u > 0$. Since $x^2 < 2$ we have $x/2 < 1/x$. Therefore, $x/2 + 1/x < 2/x$ and consequently $u > x$. Since by (2.11.2), $u \in A$ this proves that A does not have a maximum.

As A does not have a maximum $c \notin A$; that is $c^2 < 2$ is not true. As B does not have a minimum $c \notin B$; that is $c^2 > 2$ is not true. Consequently, by Axiom 12, $c^2 = 2$. □

Exercise 2.11.3. Prove that $\mathbb{Q}$ is a proper subset of $\mathbb{R}$.

Theorem 2.11.4. *Let $a, b \in \mathbb{R}$ be such that $a < b$. Let $f : \mathbb{N} \to (a, b)$ be a function. Then f is not a surjection.*

Proof. We first prove a lemma.

Lema. If $u, v \in \mathbb{R}$ and $u < v$, then

$$u < \frac{2}{3}u + \frac{1}{3}v < \frac{1}{3}u + \frac{2}{3}v < v.$$

Proof of the Lemma. By OA, $u < v$, implies the following three inequalities, $2u + u < 2u + v$, $u + (u + v) < v + (u + v)$, and $u + 2v < v + 2v$. Simplifying yields, $3u < 2u + v$, $2u + v < u + 2v$, $u + 2v < 3v$. Now OT and OM with $1/3 > 0$, produce the displayed inequalities.

Proof of Theorem. Let $\phi : \mathbb{N} \to (a, b)$ be an arbitrary function. We will prove that ϕ is not a surjection.

First we define recursively two sequences $\alpha : \mathbb{N} \to (a, b)$ and $\beta : \mathbb{N} \to (a, b)$. Here is the recursive definition:

Base step: Define

$$\alpha_1 = \frac{2}{3}\phi_1 + \frac{1}{3}b \qquad \text{and} \qquad \beta_1 = \frac{2}{3}\phi_1 + \frac{2}{3}b.$$

Recursive step: Let $n \in \mathbb{N}$ and assume that α_n and β_n are defined. Define

$$\alpha_{n+1} = \begin{cases} \frac{2}{3}\alpha_n + \frac{1}{3}\beta_n & \text{if } \ \phi_{n+1} \notin (\alpha_n, \beta_n), \\ \frac{2}{3}\phi_{n+1} + \frac{1}{3}\beta_n & \text{if } \ \phi_{n+1} \in (\alpha_n, \beta_n), \end{cases}$$

and

$$\beta_{n+1} = \begin{cases} \frac{1}{3}\alpha_n + \frac{2}{3}\beta_n & \text{if } \ \phi_{n+1} \notin (\alpha_n, \beta_n), \\ \frac{1}{3}\phi_{n+1} + \frac{2}{3}\beta_n & \text{if } \ \phi_{n+1} \in (\alpha_n, \beta_n). \end{cases}$$

Using Mathematical induction we can prove that the sequences α and β have the following five properties:

(1) For all $n \in \mathbb{N}$ we have $a < \alpha_n < \alpha_{n+1} < \beta_{n+1} < \beta_n < b$.

(2) $\alpha_k < \alpha_m$ for all $k, m \in \mathbb{N}$ such that $k < m$.

(3) $\beta_m < \beta_k$ for all $k, m \in \mathbb{N}$ such that $k < m$.

(4) $\alpha_k < \beta_m$ for all $k, m \in \mathbb{N}$.

(5) $\phi_n \notin [\alpha_n, \beta_n]$ for all $n \in \mathbb{N}$.

Proof of (1): Let $n \in \mathbb{N}$. The statement $P(n)$ is the lats part of the sentence in (1). Since $a < \phi_1 < b$, Lemma implies $a < \phi_1 < \alpha_1 < \beta_1 < b$. This proves $P(1)$.

Now prove the Inductive step. Let $n \in \mathbb{N}$. Assume that $P(n)$ is true. That is assume $a < \alpha_n < \beta_n < b$. In each case in the definition of α_{n+1} and β_{n+1} Lemma implies that $P(n + 1)$ is true. (The details are easy to check.)

The formal proofs of (2) and (3) require two inductions since the statements involve two natural numbers. But, (2) and (3) are intuitively clear consequences of (1). The statement (4) follows from OT and (1), (2) and (3).

The formal proof of (5) is an easy consequence of Lemma.

Now we are ready to use the Completeness Axiom and complete the proof that ϕ is not a surjection. Define two sets

$$A = \big\{\alpha_n : n \in \mathbb{N}\big\} \quad \text{and} \quad B = \big\{\beta_n : n \in \mathbb{N}\big\}.$$

Clearly A and B are nonempty subsets of $\mathbb{R}$. By (4) $x \leq y$ for all $x \in A$ and all $y \in B$. By CA there exists $c \in \mathbb{R}$ such that $\alpha_k \leq c \leq \beta_m$ for all $k, m \in \mathbb{N}$. In particular $c \in (a, b)$ and $c \in \big[\alpha_n, \beta_n\big]$ for all $n \in \mathbb{N}$. The last fact and (5) imply

$$c \in \big[\alpha_n, \beta_n\big] \quad \text{and} \quad \phi_n \notin \big[\alpha_n, \beta_n\big] \quad \forall\, n \in \mathbb{N}.$$

The last displayed statement yields $c \neq \phi_n$ for all $n \in \mathbb{N}$. Thus, ϕ is not a surjection. □

2.12. More on the sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$

Exercise 2.12.1. If $b > 0$, then there exists a natural number n such that $\dfrac{1}{n} < b$.

Exercise 2.12.2. If $c \in \mathbb{R}$ and $-\dfrac{1}{n} < c < \dfrac{1}{n}$ for all $n \in \mathbb{N}$, then $c = 0$.

Exercise 2.12.3. If a and b are positive real numbers, then there exists a natural number n such that $b < na$. This tells us that, even if a is quite small and b quite large, some integer multiple of a will exceed b.

Remark 2.12.4. Note that if we set $b = 1$ we obtain the statement in Exercise 2.12.1 and if we set $a = 1$, we obtain the Archimedean property.

Exercise 2.12.5. If $\alpha, \beta \in \mathbb{R}$. $0 \leq \alpha < \beta$ and $\beta - \alpha > 1$, then there exists $m \in \mathbb{N}$ such that $\alpha < m < \beta$ (that is, there exists $m \in (\alpha, \beta) \cap \mathbb{N}$).

Solution. Consider the set $A = \{k \in \mathbb{N} : \alpha < k\}$. By Exercise 2.11.1 the set A is not empty. Clearly $A \subseteq \mathbb{N}$. By Exercise 2.8.15 the set A has a minimum element. Put $m = \min A$. Now we have to prove that $\alpha < m < \beta$. Since $m \in A$, we have $\alpha < m$. In order to prove that $m < \beta$ we consider the following two cases: Case 1: $m = 1$ and Case 2: $m \in \mathbb{N}\backslash\{1\}$.
Case 1. Assume that $m = 1$. Since $1 < \beta - \alpha < \beta$ we see that $m < \beta$.
Case 2. Assume that $m \in \mathbb{N}\backslash\{1\}$. By Theorem 2.6.6 (ii), $j = m - 1 \in \mathbb{N}$. Clearly $j < m$. Is $j \in A$? NO: j is not in A since $j < m = \min A$. Since $j \in \mathbb{N}$ and $j \notin A$, we have $j \leq \alpha$. Add 1 to both sides of this inequality and we get $m = j+1 \leq \alpha+1 < \beta$. Therefore $m < \beta$. □

Remark 2.12.6. The goal of Exercise 2.12.5 is to prove the existence of a natural number with a certain property. In other words, given α and β we must construct a natural number m with the given property. What are possible tools for this construction? The proof above uses a remarkable idea how to do "constructions" of numbers:

- Step 1: Identify a set of candidates for the desired number.
- Step 2: The set of candidates is nice enough that it has an extreme element. (In this case it is a minimum.)
- Step 3: Where else could our special number be hiding?

Exercise 2.12.7. If $a, b \in \mathbb{R}$ and $a < b$, then there exists a rational number $r \in \mathbb{Q}$ such that $a < r < b$. Hint. Consider three cases: $0 \leq a < b$, $a < 0 < b$ and $a < b \leq 0$. Use 2.12.5 and 2.12.3 for the first case. The other two cases are easy.

Exercise 2.12.8. Let $a, b \in \mathbb{R}$ and $a < b$. Prove that $\{x \in \mathbb{Q} : a < x < b\}$ is an infinite set.

Exercise 2.12.9. Let $A \subset \mathbb{N}$ be a nonempty and bounded subset of $\mathbb{N}$. Prove that A is finite.

Exercise 2.12.10. A nonempty bounded below subset of $\mathbb{Z}$ has a minimum.

Exercise 2.12.11. A nonempty bounded above subset of $\mathbb{Z}$ has a maximum.

Exercise 2.12.12. Let $x \in \mathbb{R}$ be arbitrary. Prove that the set $\{k \in \mathbb{Z} : k \leq x\}$ has a maximum.

Exercise 2.12.13. Let $x \in \mathbb{R}$ be arbitrary. Prove that the set $\{k \in \mathbb{Z} : k \geq x\}$ has a minimum.

Based on the last two exercises we define the following two functions which relate real numbers to integers.

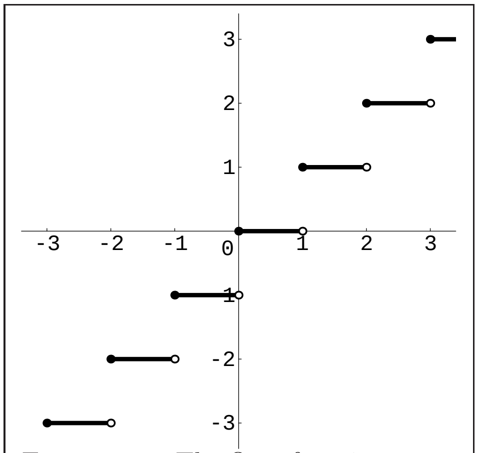


FIGURE 12. The floor function

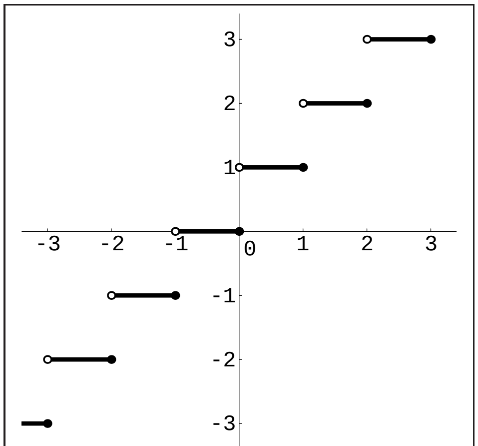


FIGURE 13. The ceiling function

Definition 2.12.14. The *floor* function is defined by

$$\operatorname{flr}(x) = \lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \leq x\}, \quad x \in \mathbb{R}.$$

The *ceiling* function is defined by

$$\operatorname{clg}(x) = \lceil x \rceil := \min\{k \in \mathbb{Z} : x \leq k\}, \quad x \in \mathbb{R}.$$

Exercise 2.12.15. Prove $x - 1 < \lfloor x \rfloor \leq x$ for all $x \in \mathbb{R}$.

Exercise 2.12.16. Prove $x \leq \lceil x \rceil < x + 1$ for all $x \in \mathbb{R}$.

Exercise 2.12.17. Prove $x \left\lceil \dfrac{1}{x} \right\rceil \geq 1$ for all $x > 0$.

Exercise 2.12.18. Let $a, b \in \mathbb{R}$ and assume $b - a \geq 1$. Prove $a < \dfrac{\lceil a \rceil + \lfloor b \rfloor}{2} < b$.

Exercise 2.12.19. Let $a, b \in \mathbb{R}$ and assume $a < b$. Prove

$$a < \frac{\left\lceil \left\lceil \dfrac{1}{b-a} \right\rceil a \right\rceil + \left\lfloor \left\lceil \dfrac{1}{b-a} \right\rceil b \right\rfloor}{2\left\lceil \dfrac{1}{b-a} \right\rceil} < b.$$

Exercise 2.12.20. If $a, b \in \mathbb{R}$ and $a < b$, then there exists a rational number $r \in \mathbb{Q}$ such that $a < r < b$

Exercise 2.12.21. Let $a, b \in \mathbb{R}$ and $a < b$. Prove that $\bigl\{x \in \mathbb{Q} : a < x < b\bigr\}$ is an infinite set.

2.13. Infimums and supremums

Definition 2.13.1. Let A be a nonempty subset of $\mathbb{R}$. A number $w \in \mathbb{R}$ is a *supremum* (or a *least upper bound*) of A if

(i) w is an upper bound for A, and
(ii) if v is an upper bound for A and $w \neq v$, then $w < v$.

Definition 2.13.2. Let A be a nonempty subset of $\mathbb{R}$. A number $u \in \mathbb{R}$ is an *infimum* (or *greatest lower bound*) of A if

(i) u is a lower bound for A, and
(ii) if v is a lower bound for A and $v \neq u$, then $v < u$.

If u and w are as in Definitions 2.13.1 and 2.13.2, we write

$$w = \sup A \ (= \operatorname{lub} A\ ) \quad \text{and} \quad u = \inf A\,(= \operatorname{glb} A).$$

Exercise 2.13.3. If $(\sup A) \in A$, then $\sup A = \max A$. State and prove the analogous statement for $\inf A$.

Exercise 2.13.4. Let A be a nonempty and bounded above subset of $\mathbb{R}$. Prove that the set of all upper bounds of A has a minimum.

The following exercise gives the standard form of the **Completeness axiom**.

Exercise 2.13.5. A nonempty subset of $\mathbb{R}$ that is bounded above has a supremum. In other words, if a set $A \subset \mathbb{R}$ is nonempty and bounded above, then $\sup A$ exists and it is a real number.

Exercise 2.13.6. A nonempty and bounded below subset of $\mathbb{R}$ has an infimum.

Exercise 2.13.7. Let $A \subset \mathbb{R}$, $A \neq \emptyset$ and A is bounded below. Prove that $a = \inf A$ if and only if

(a) a is a lower bound of A, that is, $a \leq x$, for all $x \in A$;
(b) for each $\epsilon > 0$ there exists $x \in A$ such that $x < a + \epsilon$.

Notice that x in (b) depends on ϵ. Sometimes it is useful to indicate this dependence by writing x_ϵ or $x(\epsilon)$ instead of x.

Exercise 2.13.8. State and prove a characterization of $\sup A$ which is analogous to the characterization of $\inf A$ given in Exercise 2.13.7.

Exercise 2.13.9. Find sup and inf for the sets $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ and $B = \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$. Formal proofs are required. (By a formal proof I mean a rigorous mathematical proof of properties (i) and (ii) in Definitions 2.13.1 and 2.13.2.)

Exercise 2.13.10. Find sup and inf for the set $\left\{\frac{n^{(-1)^n}}{n+1} : n \in \mathbb{N}\right\}$.

Exercise 2.13.11. Let A be a nonempty and bounded above subset of $\mathbb{R}$. If B is a nonempty subset of A, then B is bounded above and $\sup B \leq \sup A$. Formulate the corresponding statement for the infimums.

Exercise 2.13.12. Let A and B be nonempty bounded above subsets of $\mathbb{R}$. Prove

$$\sup(A \cup B) = \max\{\sup A, \sup B\}.$$

Exercise 2.13.13. Let A and B be nonempty subsets of $\mathbb{R}$ such that for all $x \in A$ and for all $y \in B$ we have $x \leq y$. Prove that $\sup A \leq \inf B$.

If the condition $x \leq y$ is replaced by the condition $x < y$, can we conclude that $\sup A < \inf B$?

Exercise 2.13.14. Suppose that A and B are nonempty subsets of $\mathbb{R}$ such that for all $x \in A$ and for all $y \in B$ we have $x \leq y$. Prove that $\sup A = \inf B$ if and only if for each $\delta > 0$ there exist $x \in A$ and $y \in B$ such that $x + \delta > y$.

Exercise 2.13.15. Let A be a nonempty and bounded above subset of $\mathbb{R}$, and let F be a finite subset of A. If $(\sup A) \notin A$, then $\sup(A \backslash F) = \sup A$.

State and prove an analogous statement for $\inf A$?

Exercise 2.13.16. Consider the set

$$A = \{x \in \mathbb{R} : x > 0 \quad \text{and } x^2 < 2\}.$$

Prove that A is nonempty and bounded. Put $\alpha = \sup A$. Prove that $\alpha^2 = 2$.

Note: Do this exercise using only the properties of the supremum. Do not use the existence of $\sqrt{2}$ proved in Exercise 2.11.2.

Exercise 2.13.17. Let $a > 0$. Consider the set

$$A = \{x \in \mathbb{R} : x > 0 \quad \text{and} \quad x^2 < a\}.$$

Prove that A is nonempty and bounded. Put $\alpha = \sup A$. Prove that $\alpha^2 = a$.

Exercise 2.13.18. Let $a > 0$ and $n \in \mathbb{N}$. Consider the set

$$A = \{x \in \mathbb{R} : x > 0 \quad \text{and} \quad x^n < a\}.$$

Prove that A is nonempty and bounded. Put $\alpha = \sup A$. Prove that $\alpha^n = a$.

Exercise 2.13.19. Let $n \in \mathbb{N}$. Prove that the function $f : [0, +\infty) \to [0, +\infty)$ defined by $f(x) = x^n, x \geq 0$, is a bijection.

Definition 2.13.20. The inverse of the bijection $f : [0, +\infty) \to [0, +\infty)$ from Exercise 2.13.19 is called the *n-th root function*. For $x \geq 0$ the value of the n-th root function at x is denoted by $\sqrt[n]{x}$ and it is called the *n-th root of x*.

Exercise 2.13.21. Let A be a nonempty subset of $\mathbb{R}$. Define the difference set A_d of A to be

$$A_d := \{b - a : a, b \in A \text{ and } a < b\}$$

If A is infinite and bounded, then $\inf A_d = 0$.

Remark 2.13.22. A partial contrapositive of the last exercise is as follows. If A is infinite and $\inf A_d > 0$, then A is not bounded. Since we proved that $\mathbb{N}$ is infinite and clearly $\mathbb{N}_d = \mathbb{N}$ and hence $\inf \mathbb{N}_d = 1$, the contrapositive of Exercise 2.13.21 implies that $\mathbb{N}$ is not bounded. This provides an alternative proof of the Archimedean property proved in Exercise 2.11.1. Notice that the existence of the floor and the ceiling function and the fact that there are rational numbers in any open interval all depend on the Archimedean property, and via the Archimedean property these properties depend on the Completeness Axiom.

In conclusion, the set $\mathbb{R}$ is completely described by Axioms 1 through 15 and the Completeness Axiom. All claims about real numbers can be proved using these 16 axioms and their consequences. As you probably already noticed in proofs we also use elementary properties of sets and operations with sets.

2.14. The topology of $\mathbb{R}$

The terminology that we introduce in the next definition provides the essential vocabulary of the modern analysis.

Definition 2.14.1. All points in this definition are elements of $\mathbb{R}$ and all sets are subsets of $\mathbb{R}$.

(a) Let $\epsilon > 0$. A *neighborhood* (or ϵ-*neighborhood*) of a point a is the set

$$N(a, \epsilon) = \{x \in \mathbb{R} : |x - a| < \epsilon\} = (a - \epsilon, a + \epsilon)$$

The number ϵ is called the *radius* of $N(a, \epsilon)$.

(b) A point a is an *accumulation point* of a set E if every neighborhood of a contains a point $x \neq a$ such that $x \in E$. That is, a is an *accumulation point* of the set E if

$$E \bigcap \big(N(a, \epsilon) \setminus \{a\}\big) \neq \emptyset \quad \text{for all} \quad \epsilon > 0.$$

(c) A set E is *closed* if it contains all its accumulation points. That is, E is closed if the following implication holds:

$$x \text{ is an accumulation point of } E \;\Rightarrow\; x \in E.$$

(d) A point a is an *interior point* of the set E if there is a neighborhood of a that is a subset of E. That is, a is an *interior point* of E if there exists $\epsilon > 0$ such that $N(a, \epsilon) \subseteq E$.

(e) A set E is *open* if every point of E is an interior point of E.

(f) A set E is *compact* if every infinite subset of E has an accumulation point in E.

(g) Let $E \subseteq F$. A set E is *dense* in F if every neighborhood of every point in F contains a point of E.

Exercise 2.14.2. Find all accumulation points of the set $\left\{\dfrac{n^{(-1)^n}}{n+1} : n \in \mathbb{N}\right\}$. Provide formal proofs.

Exercise 2.14.3. Find all accumulation points of $\left\{ \frac{4}{n} + \frac{n}{4} - \left\lfloor \frac{n}{4} \right\rfloor \; : \; n \in \mathbb{N} \right\}$. Provide formal proofs.

Exercise 2.14.4. Let $A \subset \mathbb{R}$ be a bounded set. If A does not have a maximum, then $\sup A$ is an accumulation point of A. State and prove the analogous statement for $\inf A$.

Exercise 2.14.5. Let $a < b$. Prove that the open interval (a, b) is an open set. Prove that the complement of (a, b), that is the set $\mathbb{R} \setminus (a, b)$ is closed. (Hint: State the contrapositive of the implication in the definition of a closed set. Simplify the contrapositive using the concept of an interior point.)

Exercise 2.14.6. Let $a < b$. Prove that the closed interval $[a, b]$ is a closed set. Prove that the complement of $[a, b]$, that is the set $\mathbb{R} \setminus [a, b]$ is open.

Exercise 2.14.7. Let $a < b$. Consider the interval $[a, b)$. Is this a closed set? Is it open?

Exercise 2.14.8. Is $\mathbb{R}$ a closed set? Is it open?

Exercise 2.14.9. Prove that $G \subset \mathbb{R}$ is open if and only if $\mathbb{R} \setminus G$ is closed.

Exercise 2.14.10. Let $a < b$. Prove that the closed interval $[a, b]$ is a compact set.

Hint 1: Use Proposition 2.8.10 and Exercise 2.14.4.

Hint 2: Use Cantor's intersection theorem. Consider an arbitrary infinite subset E of $[a, b]$. Define a sequence of closed intervals $[a_n, b_n]$, $n \in \mathbb{N}$, such that, for all $n \in \mathbb{N}$,

$$[a_n, b_n] \subseteq [a, b], \qquad [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n], \qquad b_n - a_n = (b - a)/2^{n-1},$$

and, most importantly, $[a_n, b_n] \cap E$ is infinite.

Definition 2.14.11. A family $\mathcal{G}$ of open sets is an *open cover* for a set E if

$$E \subseteq \bigcup \{G : G \in \mathcal{G}\}.$$

Definition 2.14.12. If every open cover of a set E has a finite subfamily that is also an open cover of E, than we say that E has the *Heine-Borel* property.

Exercise 2.14.13. Let $a < b$. Prove that the closed interval $[a, b]$ has the Heine-Borel property.

Hint: Let $\mathcal{G}$ be an arbitrary open cover of $[a, b]$. Consider the set

$$S = \left\{ x \in (a, b] \; : \; \exists\, n \in \mathbb{N} \text{ and } \exists\, G_1, \ldots, G_n \in \mathcal{G} \text{ such that } [a, x] \subseteq \bigcup_{j=1}^{n} G_j \right\}.$$

2.14.1. The structure of open sets in $\mathbb{R}$.

Definition 2.14.14. A subset $I \subseteq \mathbb{R}$ is an open interval if one of the following four conditions is satisfied

- $I = \mathbb{R}$.
- There exists $a \in \mathbb{R}$ such that $I = (-\infty, a)$.
- There exists $b \in \mathbb{R}$ such that $I = (b, +\infty)$.
- There exist $a, b \in \mathbb{R}$ such that $a < b$ and $I = (a, b)$.

Exercise 2.14.15. Let $\mathcal{I}$ be an infinite family of open mutually disjoint intervals. (Mutually disjoint means that if $I_1, I_2 \in \mathcal{I}$ and $I_1 \neq I_2$, then $I_1 \cap I_2 \neq \emptyset$.) Prove that $\mathcal{I}$ is countable.

Exercise 2.14.16. Let G be a nonempty open subset of $\mathbb{R}$. Assume that $\mathbb{R} \setminus G$ is neither bounded above nor below. Prove that for each $x \in G$ there exist $a, b \in \mathbb{R} \setminus G$ such that $a < b$, $x \in (a, b)$ and $(a, b) \subseteq G$.

Exercise 2.14.17. Let G be a nonempty open subset of $\mathbb{R}$. Assume that $\mathbb{R} \setminus G$ is neither bounded above nor below. Prove that there exists a finite or countable family of open mutually disjoint intervals whose union equals G.

Exercise 2.14.18. Let G be a nonempty open subset of $\mathbb{R}$. Then there exists a finite or countable family of open mutually disjoint intervals whose union equals G.

CHAPTER 3

Sequences in $\mathbb{R}$

3.1. Definitions and examples

Definition 3.1.1. A *sequence in* $\mathbb{R}$ is a function whose domain is $\mathbb{N}$ and whose range is in $\mathbb{R}$.

Let $s : \mathbb{N} \to \mathbb{R}$ be a sequence in $\mathbb{R}$. Then the values of s are

$$s(1),\ s(2),\ s(3),\ \dots,\ s(n),\ \dots\ .$$

It is customary to write s_n instead of $s(n)$ for the values of a sequence. Sometimes a sequence will be specified by listing its first few terms

$$s_1,\ s_2,\ s_3,\ s_4,\ \dots\ ,$$

and sometimes by listing of all its terms $\{s_n\}_{n=1}^{\infty}$ or $\{s_n\}$. One way of specifying a sequence is to give a formula, or a recursion formula for its $n-$th term s_n.

Remark 3.1.2. In the above notation s is the "name" of the sequence and $n \in \mathbb{N}$ is the independent variable.

Remark 3.1.3. Notice the difference between the following two expressions:

$\{s_n\}_{n=1}^{\infty}$ This expression denotes a function (sequence).

$\{s_n\ :\ n \in \mathbb{N}\}$ This expression denotes a set: The range of a sequence $\{s_n\}_{n=1}^{\infty}$.

For example $\{1-(-1)^n\}_{n=1}^{\infty}$ stands for the function $n \mapsto 1-(-1)^n$, $n \in \mathbb{N}$, while

$$\{1-(-1)^n : n \in \mathbb{N}\} = \{0,2\}.$$

Example 3.1.4. Here we give examples of sequences given by a formula. In each formula below $n \in \mathbb{N}$.

$$\begin{array}{llllllll}
\text{(a)} & n, & \text{(b)} & n^2, & \text{(c)} & \sqrt{n}, & \text{(d)} & (-1)^n, \\
\text{(e)} & \dfrac{1}{n}, & \text{(f)} & \dfrac{1}{n^2}, & \text{(g)} & \dfrac{1}{\sqrt{n}}, & \text{(h)} & 1-\dfrac{(-1)^n}{n}, \\
\text{(i)} & \dfrac{1}{n!}, & \text{(j)} & 2^{1/n}, & \text{(k)} & n^{1/n}, & \text{(l)} & n^{(-1)^n}, \\
\text{(m)} & \dfrac{9^n}{n!}, & \text{(n)} & \dfrac{(-1)^{n+1}}{2n-1}, & \text{(o)} & \dfrac{n^{(-1)^n}}{n+1}, & \text{(p)} & \left(\dfrac{e}{n}\right)^n \dfrac{n!}{\sqrt{n}}.
\end{array}$$

Example 3.1.5. Few more sequences given by a formula are

$$\text{(a)}\ \left\{\sqrt{n^2+1}-n\right\}_{n=1}^{\infty}, \quad \text{(b)}\ \left\{\sqrt{n^2+n}-n\right\}_{n=1}^{\infty}, \quad \text{(c)}\ \left\{\sqrt{n+1}-\sqrt{n}\right\}_{n=1}^{\infty}.$$

Example 3.1.6. In this example we give several recursively defined sequences.

(a) $s_1 = 1$ and $(\forall n \in \mathbb{N})$ $s_{n+1} = -\dfrac{s_n}{2}$,

(b) $x_1 = 1$ and $(\forall n \in \mathbb{N})$ $x_{n+1} = 1 + \dfrac{x_n}{4}$,

(c) $x_1 = 2$ and $(\forall n \in \mathbb{N})$ $x_{n+1} = \dfrac{x_n}{2} + \dfrac{1}{x_n}$,

(d) $a_1 = \sqrt{2}$ and $(\forall n \in \mathbb{N})$ $a_{n+1} = \sqrt{2 + a_n}$,

(e) $s_1 = 1$ and $(\forall n \in \mathbb{N})$ $s_{n+1} = \sqrt{1 + s_n}$,

(f) $x_1 = 0$ and $(\forall n \in \mathbb{N})$ $x_{n+1} = \dfrac{9 + x_n}{10}$.

For a recursively defined sequence it is useful to evaluate the values of the first few terms to get an idea how sequence behaves.

Example 3.1.7. The most important examples of sequences are listed below:

(3.1.1) $b_n = a, \ n \in \mathbb{N}, \quad \text{where} \quad a \in \mathbb{R},$

(3.1.2) $p_n = a^n, \ n \in \mathbb{N}, \quad \text{where} \quad -1 < a < 1,$

(3.1.3) $E_n = \left(1 + \dfrac{1}{n}\right)^n, \quad n \in \mathbb{N},$

(3.1.4) $G_1 = a + ax \ \text{ and } \ (\forall\, n \in \mathbb{N})\ G_{n+1} = G_n + ax^{n+1}, \ \text{where} \ -1 < x < 1,$

(3.1.5) $S_1 = 2 \ \text{ and } \ (\forall\, n \in \mathbb{N})\ S_{n+1} = S_n + \dfrac{1}{(n+1)!},$

(3.1.6) $v_1 = 1 + a \ \text{ and } \ (\forall\, n \in \mathbb{N})\ v_{n+1} = v_n + \dfrac{a^{n+1}}{(n+1)!}, \quad \text{where} \quad a \in \mathbb{R}.$

Definition 3.1.8. Let $\{a_n\}$ be a sequence in $\mathbb{R}$. A sequence which is recursively defined by

(3.1.7) $$S_1 = a_1 \ \text{ and } \ (\forall\, n \in \mathbb{N})\ S_{n+1} = S_n + a_{n+1},$$

is called a *sequence of partial sum* corresponding to $\{a_n\}$.

Example 3.1.9. The sequences of partial sums associated with the sequences in Example 3.1.4 (e), (f) and (n) are important examples for Definition 3.1.8. Notice also that the sequences in (3.1.4), (3.1.5) and (3.1.6) are sequences of partial sums. All of these are very important.

3.2. Bounded sequences

Definition 3.2.1. Let $\{s_n\}$ be a sequence in $\mathbb{R}$.

(1) If a real number M satisfies

$$s_n \leq M \quad \text{for all} \quad n \in \mathbb{N}$$

then M is called an *upper bound* of $\{s_n\}$ and the sequence $\{s_n\}$ is said to be *bounded above*.

(2) If a real number m satisfies

$$m \leq s_n \quad \text{for all} \quad n \in \mathbb{N},$$

then m is called a *lower bound* of $\{s_n\}$ and the sequence $\{s_n\}$ is said to be *bounded below*.

(3) The sequence $\{s_n\}$ is said to be *bounded* if it is bounded above and bounded below.

Remark 3.2.2. Clearly, a sequence $\{s_n\}$ is bounded above if and only if the set $\{s_n : n \in \mathbb{N}\}$ is bounded above. Similarly, a sequence $\{s_n\}$ is bounded below if and only if the set $\{s_n : n \in \mathbb{N}\}$ is bounded below.

Remark 3.2.3. The sequence $\{s_n\}$ is bounded if and only if there exists a real number $K > 0$ such that $|s_n| \leq K$ for all $n \in \mathbb{N}$.

Exercise 3.2.4. There is a huge task here. For each sequence given in this section it is of interest to determine whether it is bounded or not. As usual, some of the proofs are easy, some are hard. It is important to do few easy proofs and observe their structure. This will provide the setting to appreciate proofs for hard examples.

3.3. The definition of a convergent sequence

Definition 3.3.1. A sequence $\{s_n\}$ is a *constant* sequence if there exists $L \in \mathbb{R}$ such that $s_n = L$ for all $n \in \mathbb{N}$.

Exercise 3.3.2. Prove that the sequence $s_n = \left\lfloor \dfrac{3n-1}{2n} \right\rfloor$, $n \in \mathbb{N}$, is a constant sequence.

Definition 3.3.3. A sequence $\{s_n\}$ is *eventually constant* if there exists $L \in \mathbb{R}$ and $N_0 \in \mathbb{R}$ such that $s_n = L$ for all $n \in \mathbb{N}$, $n > N_0$.

Exercise 3.3.4. Prove that the sequence $s_n = \left\lceil \dfrac{3n-2}{2n+3} \right\rceil$, $n \in \mathbb{N}$, is eventually constant.

Exercise 3.3.5. Prove that the sequence $s_n = \left\lfloor \dfrac{5n-(-1)^n}{n/2+5} \right\rfloor$, $n \in \mathbb{N}$, is eventually constant.

Definition 3.3.6. A sequence $\{s_n\}$ *converges* if there exists $L \in \mathbb{R}$ such that for each $\epsilon > 0$ there exists a real number $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \qquad \Rightarrow \qquad |s_n - L| < \epsilon.$$

The number L is called the *limit* of the sequence $\{s_n\}$. We also say that $\{s_n\}$ *converges to* L and write

$$\lim_{n\to\infty} s_n = L \qquad \text{or} \qquad s_n \to L \;\; (n \to \infty).$$

If a sequence does not converge we say that it *diverges*.

Remark 3.3.7. The definition of convergence is a complicated statement. Formally it can be written as:

$$\exists\, L \in \mathbb{R} \;\text{ s.t. }\; \forall\, \epsilon > 0 \;\; \exists\, N(\epsilon) \in \mathbb{R} \;\text{ s.t. }\; \forall\, n \in \mathbb{N}, \;\; n > N(\epsilon) \;\Rightarrow\; |s_n - L| < \epsilon.$$

Exercise 3.3.8. State the negation of the statement in remark 3.3.7.

3.3.1. My informal discussion of convergence. It is easy to agree that the constant sequences are simplest possible sequences. For example the sequence

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
c_n	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

or formally, $c_n = 1$ for all $n \in \mathbb{N}$, is a very simple sequence. No action here! In this case, clearly, $\lim_{n\to\infty} c_n = 1$.

Now, I define $s_n = \dfrac{n-(-1)^n}{n}$, $n \in \mathbb{N}$, and I ask: Is $\{s_n\}$ a constant sequence? Just looking at the first few terms

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
s_n	2	$\frac{1}{2}$	$\frac{4}{3}$	$\frac{3}{4}$	$\frac{6}{5}$	$\frac{5}{6}$	$\frac{8}{7}$	$\frac{7}{8}$	$\frac{10}{9}$	$\frac{9}{10}$	$\frac{12}{11}$	$\frac{11}{12}$	$\frac{14}{13}$	$\frac{13}{14}$	$\frac{16}{15}$	$\frac{15}{16}$	$\frac{18}{17}$

indicates that this sequence is not constant. The table above also indicates that the sequence $\{s_n\}$ is not eventually constant. But imagine that you have a calculator which is capable of displaying only one decimal place. On this calculator the first terms of this sequence would look like:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
s_n	2.0	0.5	1.3	0.8	1.2	0.8	1.1	0.9	1.1	0.9	1.1	0.9	1.1	0.9	1.1

and the next 15 terms would look like:

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
s_n	0.9	1.1	0.9	1.1	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Basically, after the 20-th term this calculator does not distinguish s_n from 1. That is, this calculator leads us to think that $\{s_n\}$ is eventually constant. Why is this? On this calculator all numbers between $0.95 = 1 - 1/20$ and $1.05 = 1 + 1/20$ are represented as 1, and for our sequence we can prove that

$$n \in \mathbb{N}, \quad n > 20 \qquad \Rightarrow \qquad 1 - \frac{1}{20} < s_n < 1 + \frac{1}{20},$$

or, equivalently,

$$n \in \mathbb{N}, \quad n > 20 \qquad \Rightarrow \qquad \bigl|s_n - 1\bigr| < \frac{1}{20}.$$

In the notation of Definition 3.3.6 this means $N(1/20) = 20$.

One can reasonably object that the above calculator is not very powerful and propose to use a calculator that can display three decimal places. Then the terms of $\{s_n\}$ starting with $n = 21$ are

n	21	22	23	24	25	26	27	28	29	30
s_n	1.048	0.955	1.043	0.958	1.040	0.962	1.037	0.964	1.034	0.967

Now the question is: Can we fool this powerful calculator to think that $\{s_n\}$ is eventually constant? Notice that on this calculator all numbers between $0.9995 = 1 - 1/2000$ and $1.0005 = 1 + 1/2000$ are represent as 1. Therefore, in the notation of Definition 3.3.6, we need $N(1/2000)$ such that

$$n \in \mathbb{N}, \quad n > N(1/2000) \qquad \Rightarrow \qquad 1 - \frac{1}{2000} < s_n < 1 + \frac{1}{2000}.$$

An easy calculation shows that $N(1/2000) = 2000$. That is

$$n \in \mathbb{N}, \quad n > 2000 \qquad \Rightarrow \qquad \bigl|s_n - 1\bigr| < \frac{1}{2000}.$$

This is illustrated by the following table

n	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005
s_n	0.999	1.001	0.999	1.001	1.000	1.000	1.000	1.000	1.000	1.000

Hence, even this more powerful calculator is fooled into thinking that $\{s_n\}$ is eventually constant.

In computer science the precision of a computer is measured by the number called the *machine epsilon* (also called *macheps, machine precision* or *unit round-off*). It is the smallest number that gives a number greater than 1 when added to 1.

Now, Definition 3.3.6 can be paraphrased as: A sequence converges if on each computer it appears to be eventually constant. This is the reason why I think that instead of the phrase “a sequence is convergent” we could use the phrase “a sequence is constantish.”

3.4. Finding $N(\epsilon)$ for a convergent sequence

Example 3.4.1. Prove that $\displaystyle\lim_{n\to\infty} \frac{2n-1}{n+3} = 2$.

Solution. We prove the given equality using Definition 3.3.6. To do that for each $\epsilon > 0$ we have to find $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \qquad \Rightarrow \qquad \left|\frac{2n-1}{n+3} - 2\right| < \epsilon. \tag{3.4.1}$$

Let $\epsilon > 0$ be given. We can think of n as an unknown in $\left|\dfrac{2n-1}{n+3} - 2\right| < \epsilon$ and solve this inequality for n. To this end first simplify the left-hand side:

$$\left|\frac{2n-1}{n+3} - 2\right| = \left|\frac{2n-1-2n-6}{n+3}\right| = \frac{|-7|}{|n+3|} = \frac{7}{n+3}. \tag{3.4.2}$$

Now, $\dfrac{7}{n+3} < \epsilon$ is much easier to solve for $n \in \mathbb{N}$:

$$\frac{7}{n+3} < \epsilon \quad \Leftrightarrow \quad \frac{n+3}{7} > \frac{1}{\epsilon} \quad \Leftrightarrow \quad n+3 > \frac{7}{\epsilon} \quad \Leftrightarrow \quad n > \frac{7}{\epsilon} - 3. \tag{3.4.3}$$

Now (3.4.3) indicates that we can choose $N(\epsilon) = \frac{7}{\epsilon} - 3$.

Now we have $N(\epsilon)$, but to complete the formal proof, we have to prove implication (3.4.1). The proof follows. Let $n \in \mathbb{N}$ and $n > \frac{7}{\epsilon} - 3$. Then the equivalences in (3.4.3) imply that $\frac{7}{n+3} < \epsilon$. Since by (3.4.3), $\left|\frac{2n-1}{n+3} - 2\right| = \frac{7}{n+3}$, it follows that $\left|\frac{2n-1}{n+3} - 2\right| < \epsilon$. This completes the proof of implication (3.4.1). $\square$

Remark 3.4.2. This remark is essential for the understanding of the process described in the following examples. In the solution of Example 3.4.1 we found (in some sense) the smallest possible $N(\epsilon)$. It is important to notice that implication (3.4.1) holds with any larger value for “$N(\epsilon)$.” For example, implication (3.4.1) holds if we set $N(\epsilon) = \frac{7}{\epsilon}$. With this new $N(\epsilon)$ we can prove implication (3.4.1) as follows. Let $n \in \mathbb{N}$ and $n > \frac{7}{\epsilon}$. Then $\frac{7}{n} < \epsilon$. Since clearly $\frac{7}{n+3} < \frac{7}{n}$, the last two inequalities imply that $\frac{7}{n+3} < \epsilon$ and we can continue with the same proof as in the solution of Example 3.4.1.

Example 3.4.3. Prove that $\displaystyle\lim_{n\to\infty} \frac{1}{n^3 - n + 1} = 0.$

Solution. We prove the given equality using Definition 3.3.6. To do that for each $\epsilon > 0$ we have to find $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \quad \Rightarrow \quad \left| \frac{1}{n^3 - n + 1} - 0 \right| < \epsilon. \tag{3.4.4}$$

Let $\epsilon > 0$ be given. We can think of n as an unknown in $\left| \dfrac{1}{n^3 - n + 1} - 0 \right| < \epsilon$ and solve this inequality for n. To this end first simplify the left-hand side:

$$\left| \frac{1}{n^3 - n + 1} - 0 \right| = \left| \frac{1}{n^3 - n + 1} \right| = \frac{|1|}{|n^3 - n + 1|} = \frac{1}{n^3 - n + 1}. \tag{3.4.5}$$

Unfortunately $\dfrac{1}{n^3 - n + 1} < \epsilon$ is not easy to solve for $n \in \mathbb{N}$. Therefore we use the idea from Remark 3.4.2 and replace the quantity $\dfrac{1}{n^3 - n + 1}$ with a larger quantity. To make a fraction larger we have to make the denominator smaller. Notice that $n^2 - n = n(n-1) \geq n - 1$ for all $n \in \mathbb{N}$. Therefore for all $n \in \mathbb{N}$ we have

$$n^3 - n + 1 = n^3 - (n-1) \geq n^3 - n(n-1) = n(n^2 - n + 1) \geq n.$$

Consequently,

$$\frac{1}{n^3 - n + 1} \leq \frac{1}{n}. \tag{3.4.6}$$

Now, $\dfrac{1}{n} < \epsilon$ is truly easy to solve for $n \in \mathbb{N}$:

$$\frac{1}{n} < \epsilon \quad \Leftrightarrow \quad n > \frac{1}{\epsilon}. \tag{3.4.7}$$

Hence we set $N(\epsilon) = \frac{1}{\epsilon}$.

Now we have $N(\epsilon)$, but to complete the formal proof, we have to prove implication (3.4.4). The proof follows. Let $n \in \mathbb{N}$ and $n > \frac{1}{\epsilon}$. Then the equivalence in (3.4.7) implies that $\frac{1}{n} < \epsilon$. By (3.4.6), $\frac{1}{n^3-n+1} \leq \frac{1}{n}$. The last two inequalities yield that $\frac{1}{n^3-n+1} < \epsilon$. By (3.4.5) it follows that $\left| \frac{1}{n^3-n+1} - 0 \right| < \epsilon$. This completes the proof of implication (3.4.4). □

Example 3.4.4. Prove that $\displaystyle\lim_{n\to\infty} \frac{n^2 - 1}{n^2 - 2n + 2} = 1.$

Solution. We prove the given equality using Definition 3.3.6. To do that for each $\epsilon > 0$ we have to find $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \quad \Rightarrow \quad \left| \frac{n^2 - 1}{n^2 - 2n + 2} - 1 \right| < \epsilon. \tag{3.4.8}$$

Let $\epsilon > 0$ be given. We can think of n as an unknown in $\left|\dfrac{n^2-1}{n^2-2n+2} - 1\right| < \epsilon$ and solve this inequality for n. To this end first simplify the left-hand side:

$$\left|\frac{n^2-1}{n^2-2n+2} - 1\right| = \left|\frac{n^2-1-n^2+2n-2}{n^2-2n+2}\right| = \frac{|2n-3|}{n^2-2n+2}. \tag{3.4.9}$$

Unfortunately $\dfrac{|2n-3|}{n^2-2n+2} < \epsilon$ is not easy to solve for $n \in \mathbb{N}$. Therefore we use the idea from Remark 3.4.2 and replace the quantity $\dfrac{|2n-3|}{n^2-2n+2}$ with a larger quantity. Here is one way to discover a desired inequality. We first notice that for all $n \in \mathbb{N}$ the following two inequalities hold

$$|2n-3| \leq 2n \tag{3.4.10}$$

and

$$n^2-2n+2 = \frac{n^2}{2} + \frac{1}{2}\bigl(n^2-4n+4\bigr) = \frac{n^2}{2} + \frac{1}{2}(n-2)^2 \geq \frac{n^2}{2}. \tag{3.4.11}$$

Consequently

$$\frac{|2n-3|}{n^2-2n+2} \leq \frac{2n}{n^2/2} = \frac{4}{n}. \tag{3.4.12}$$

Now, $\dfrac{4}{n} < \epsilon$ is truly easy to solve for $n \in \mathbb{N}$:

$$\frac{4}{n} < \epsilon \qquad \Leftrightarrow \qquad n > \frac{4}{\epsilon}. \tag{3.4.13}$$

Hence we set $N(\epsilon) = \frac{4}{\epsilon}$.

Finally we have $N(\epsilon)$. But to complete the formal proof we have to prove implication (3.4.8). The proof follows. Let $n \in \mathbb{N}$ and $n > \frac{4}{\epsilon}$. Then the equivalence in (3.4.13) implies $\frac{4}{n} < \epsilon$. By (3.4.12), $\frac{|2n-3|}{n^2-2n+2} \leq \frac{4}{n}$. The last two inequalities yield $\frac{|2n-3|}{n^2-2n+2} < \epsilon$. By (3.4.9) it follows that $\left|\frac{n^2-1}{n^2-2n+2} - 1\right| < \epsilon$. This completes the proof of implication (3.4.8). □

Remark 3.4.5. For most sequences $\{s_n\}$ a proof of $\lim\limits_{n\to\infty} s_n = L$ based on Definition 3.3.6 should consist from the following steps.

(1) Use algebra to simplify the expression $|s_n - L|$. It is desirable to eliminate the absolute value.
(2) Discover an inequality of the form

$$|s_n - L| \leq b(n) \quad \text{valid for all} \quad n \in \mathbb{N}. \tag{3.4.14}$$

Here $b(n)$ should be a simple function with the following properties:
   (a) $b(n) > 0$ for all $n \in \mathbb{N}$.
   (b) $\lim\limits_{n\to\infty} b(n) = 0$. (Just check this property "mentally.")
   (c) $b(n) < \epsilon$ is easily solvable for n for every $\epsilon > 0$. The solution should be of the form "$n >$ some expression involving ϵ, call it $N(\epsilon)$."
(3) Use inequality (3.4.14) to prove the implication $n \in \mathbb{N}$, $n > N(\epsilon) \Rightarrow |s_n - L| < \epsilon$.

Exercise 3.4.6. Determine the limits (if they exist) of the sequences (e), (f), (g), (h), (i), and (n) in Example 3.1.4. Prove your claims.

Exercise 3.4.7. Determine whether the sequence $\left\{\dfrac{3n+1}{7n-4}\right\}_{n=1}^{\infty}$ converges and, if it converges, give its limit. Provide a formal proof.

Exercise 3.4.8. Determine the limits (if they exist) of the sequences in Example 3.1.5. Prove your claims.

3.5. Two standard sequences

Exercise 3.5.1. Let $a \in \mathbb{R}$ be such that $-1 < a < 1$.

(1) Prove that for all $n \in \mathbb{N}$ we have
$$|a|^n \leq \frac{1}{n\,(1-|a|)}.$$

(2) Prove that
$$\lim_{n\to\infty} a^n = 0.$$

Exercise 3.5.2. Let a be a positive real number. Prove that
$$\lim_{n\to\infty} a^{1/n} = 1.$$

Solution. Let $a > 0$. If $a = 1$, then $a^{1/n} = 1$ for all $n \in \mathbb{N}$. Therefore $\lim_{n\to\infty} a^{1/n} = 1$.

Assume $a > 1$. Then $a^{1/n} > 1$. We shall prove that

$$a^{1/n} - 1 \leq a\,\frac{1}{n} \quad \bigl(\forall\, n \in \mathbb{N}\bigr). \tag{3.5.1}$$

Put $x = a^{1/n} - 1 > 0$. Then, by Bernoulli's inequality we get
$$a = (1+x)^n \geq 1 + nx.$$

Consequently, solving for x we get that $x = a^{1/n} - 1 \leq (a-1)/n$. Since $a - 1 < a$, (3.5.1) follows.

Assume $0 < a < 1$. Then $1/a > 1$. Therefore, by already proved (3.5.1), we have
$$\left(\frac{1}{a}\right)^{1/n} - 1 \leq \frac{1}{a}\,\frac{1}{n} \quad \bigl(\forall\, n \in \mathbb{N}\bigr).$$

Since $(1/a)^{1/n} = 1/\bigl(a^{1/n}\bigr)$, simplifying the last inequality, together with the inequality $a^{1/n} < 1$, yields

$$1 - a^{1/n} \leq \frac{a^{1/n}}{a}\,\frac{1}{n} \leq \frac{1}{a}\,\frac{1}{n} \quad \bigl(\forall\, n \in \mathbb{N}\bigr). \tag{3.5.2}$$

As $a < a + 1/a$ and $1/a < a + 1/a$, the inequalities (3.5.1) and (3.5.2) imply

$$\bigl|a^{1/n} - 1\bigr| \leq \left(a + \frac{1}{a}\right)\frac{1}{n} \quad \bigl(\forall\, n \in \mathbb{N}\bigr). \tag{3.5.3}$$

Let $\epsilon > 0$ be given. Solving $\bigl(a + 1/a\bigr)\frac{1}{n} < \epsilon$ for n, reveals $N(\epsilon)$:
$$N(\epsilon) = \left(a + \frac{1}{a}\right)\frac{1}{\epsilon}$$

Now it is easy to prove the implication (Do it as an exercise!)
$$n \in \mathbb{N}, \quad n > \left(a + \frac{1}{a}\right)\frac{1}{\epsilon} \quad \Rightarrow \quad |a^{1/n} - 1| < \epsilon.$$
□

3.6. Non-convergent sequences

Exercise 3.6.1. Prove that the sequence (d) in Example 3.1.4 does not converge. Use Remark 3.3.7 and Exercise 3.3.8

Exercise 3.6.2. (Prove or Disprove) If $\{s_n\}$ does not converge to L, then there exist $\epsilon > 0$ and $N(\epsilon)$ such that $|s_n - L| \geq \epsilon$ for all $n \geq N(\epsilon)$.

3.7. Convergence and boundedness

Exercise 3.7.1. Consider the following two statements:
(A) The sequence $\{s_n\}$ is bounded.
(B) The sequence $\{s_n\}$ converges.
Is (A)⇒(B) true or false? Is (B)⇒(A) true or false? Justify your answers.

3.8. Algebra of limits of convergent sequences

Exercise 3.8.1. Let $\{s_n\}$ be a sequence in $\mathbb{R}$ and let $L \in \mathbb{R}$. Set $t_n = s_n - L$ for all $n \in \mathbb{N}$.

Prove that $\{s_n\}$ converges to L if and only if $\{t_n\}$ converges to 0.

Exercise 3.8.2. Let $c \in \mathbb{R}$. If $\lim_{n\to\infty} x_n = X$ and $z_n = c\,x_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} z_n = c\,X$.

Exercise 3.8.3. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathbb{R}$. Assume
(a) $\{x_n\}$ converges to 0,
(b) $\{y_n\}$ is bounded,
(c) $z_n = x_n y_n$ for all $n \in \mathbb{N}$.
Prove that $\{z_n\}$ converges to 0.

Exercise 3.8.4. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathbb{R}$. Assume
(a) $\lim_{n\to\infty} x_n = X$,
(b) $\lim_{n\to\infty} y_n = Y$,
(c) $z_n = x_n + y_n$ for all $n \in \mathbb{N}$.
Prove that $\lim_{n\to\infty} z_n = X + Y$.

Exercise 3.8.5. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathbb{R}$. Assume
(a) $\lim_{n\to\infty} x_n = X$,
(b) $\lim_{n\to\infty} y_n = Y$,
(c) $z_n = x_n\,y_n$ for all $n \in \mathbb{N}$.
Prove that $\lim_{n\to\infty} z_n = X\,Y$.

Exercise 3.8.6. If $\lim_{n\to\infty} x_n = X$ and $X > 0$, then there exists a real number N such that $n \geq N$ implies $x_n \geq X/2$.

Exercise 3.8.7. Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Assume
(a) $x_n \neq 0$ for all $n \in \mathbb{N}$,
(b) $\lim_{n\to\infty} x_n = X$,
(c) $X > 0$,

(d) $w_n = \dfrac{1}{x_n}$ for all $n \in \mathbb{N}$.

Prove that $\lim_{n\to\infty} w_n = \dfrac{1}{X}$.

Exercise 3.8.8. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathbb{R}$. Assume

(a) $x_n \neq 0$ for all $n \in \mathbb{N}$,
(b) $\lim_{n\to\infty} x_n = X$,
(c) $\lim_{n\to\infty} y_n = Y$,
(d) $X \neq 0$,
(e) $z_n = \dfrac{y_n}{x_n}$ for all $n \in \mathbb{N}$.

Prove that $\lim_{n\to\infty} z_n = \dfrac{Y}{X}$. (Hint: Use previous exercises.)

Exercise 3.8.9. Prove that $\lim_{n\to\infty} \dfrac{2n^2+n-5}{n^2+2n+2} =$ (insert correct value) by using the results we have proved (Exercises 3.8.2, 3.8.4, 3.8.5, 3.8.7, 3.8.8) and a small trick. You may use Definition 3.3.6 of convergence directly in this problem only to evaluate limit of the special form $\lim_{n\to\infty} \dfrac{1}{n}$.

Remark 3.8.10. The point of Exercise 3.8.9 is to see that the general properties of limits (Exercises 3.8.2, 3.8.4, 3.8.5, 3.8.7, 3.8.8) can be used to reduce complicated situations to a few simple ones, so that when the few simple ones have been done it is no longer necessary to go back to Definition 3.3.6 of convergence every time.

3.9. Convergent sequences and the order in $\mathbb{R}$

Exercise 3.9.1. Let $\{s_n\}$ be a sequence in $\mathbb{R}$. Assume

(a) $\lim_{n\to\infty} s_n = L$.
(b) There exists a real number N_0 such that $s_n \geq 0$ for all $n \in \mathbb{N}$ such that $n > N_0$.

Prove that $L \geq 0$.

Exercise 3.9.2. Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$. Assume

(a) $\lim_{n\to\infty} a_n = K$.
(b) $\lim_{n\to\infty} b_n = L$.
(c) There exists a real number N_0 such that $a_n \leq b_n$ for all $n \in \mathbb{N}$ such that $n > N_0$.

Prove that $K \leq L$.

Exercise 3.9.3. Is the following refinement of Exercise 3.9.1 true? If $\{s_n\}$ converges to L and if $s_n > 0$ for all $n \in \mathbb{N}$, then $L > 0$.

Exercise 3.9.4. Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Assume

(a) $x_n \geq 0$ for all $n \in \mathbb{N}$,
(b) $\lim_{n\to\infty} x_n = X$,
(c) $w_n = \sqrt{x_n}$ for all $n \in \mathbb{N}$.

Prove that $\lim_{n\to\infty} w_n = \sqrt{X}$.

3.10. Squeeze theorem for convergent sequences

Exercise 3.10.1. There are three sequences in this exercise: $\{a_n\}$, $\{b_n\}$ and $\{s_n\}$. Assume the following

(1) The sequence $\{a_n\}$ converges to L.
(2) The sequence $\{b_n\}$ converges to L.
(3) There exists a real number n_0 such that

$$a_n \leq s_n \leq b_n \quad \text{for all} \ \ n \in \mathbb{N}, \ n > n_0.$$

Prove that $\{s_n\}$ converges to L.

Exercise 3.10.2. (1) Let $x \geq 0$ and $n \in \mathbb{N}$. Prove the inequality

$$(1+x)^n \geq 1 + n\,x + \frac{n(n-1)}{2}\,x^2.$$

(2) Prove that for all $n \in \mathbb{N}$ we have $1 \leq n^{1/n} \leq 1 + \dfrac{2}{\sqrt{n}}$.
Hint: Apply the inequality proved in (1) to $\bigl(1 + 2/\sqrt{n}\bigr)^n$.
(3) Prove that the sequence $\{n^{1/n}\}$ converges and determine its limit.

Exercise 3.10.3. (1) Prove that $(n!)^2 \geq n^n$ for all $n \in \mathbb{N}$. Hint: Write

$$\bigl(n!\bigr)^2 = \bigl(1\cdot n\bigr)\bigl(2\cdot(n-1)\bigr)\cdots\bigl((n-1)\cdot 2\bigr)\bigl(n\cdot 1\bigr) = \prod_{k=1}^{n} k\bigl(n-k+1\bigr).$$

Then prove $k\bigl(n-k+1\bigr) \geq n$ for all $k = 1, \ldots, n$.
(2) Prove that

$$\lim_{n\to\infty} \frac{1}{\bigl(n!\bigr)^{1/n}} = 0.$$

3.11. The monotonic convergence theorem

Definition 3.11.1. A sequence $\{s_n\}$ of real numbers is said to be *non-decreasing* if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$, *strictly increasing* if $s_n < s_{n+1}$ for all $n \in \mathbb{N}$, *non-increasing* if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$, *strictly decreasing* if $s_n > s_{n+1}$ for all $n \in \mathbb{N}$. A sequence with any of these properties is said to be *monotonic*.

Exercise 3.11.2. Again a huge task here. Which of the sequences in Examples 3.1.4, 3.1.5, and 3.1.6 are monotonic? Find few monotonic ones in each example. Provide rigorous proofs.

Exercise 3.11.3. (Prove or Disprove) If $\{x_n\}$ is non-increasing, then $\{x_n\}$ converges.

The following two exercises give powerful tools for establishing convergence of a sequence.

Exercise 3.11.4. If $\{s_n\}$ is non-increasing and bounded below, then $\{s_n\}$ converges.

Exercise 3.11.5. If $\{s_n\}$ is non-decreasing and bounded above, then $\{s_n\}$ converges.

PROOF. Assume that the sequence $\{s_n\}$ is non-decreasing and bounded above. Consider the range of the sequence $\{s_n\}$. That is consider the set

$$A = \{s_n \ : \ n \in \mathbb{N}\}.$$

The set A is nonempty and bounded above. Therefore $\sup A$ exists. Put $L = \sup A$.

We will prove that $s_n \to L$ $(n \to \infty)$. Let $\epsilon > 0$ be arbitrary. Since $L = \sup A$ we have

(1) $L \geq s_n$ for all $n \in \mathbb{N}$.
(2) There exists $a_\epsilon \in A$ such that $L - \epsilon < a_\epsilon$.

Since $a_\epsilon \in A$, there exists $N_\epsilon \in \mathbb{N}$ such that $a_\epsilon = s_{N_\epsilon}$. It remains to prove that

$$n \in \mathbb{N}, \quad n > N_\epsilon \quad \Rightarrow \quad |s_n - L| < \epsilon. \tag{3.11.1}$$

Let $n \in \mathbb{N},\ n > N_\epsilon$ be arbitrary. Since we assume that $\{s_n\}$ is non-decreasing, it follows that $s_n \geq s_{N_\epsilon}$. Since $L - \epsilon < a_\epsilon = s_{N_\epsilon} \leq s_n$, we conclude that $L - s_n < \epsilon$. Since $L \geq s_n$, we have $|s_n - L| = L - s_n < \epsilon$. The implication (3.11.1) is proved. ☐

Exercise 3.11.6. There is a huge task here. Consider the sequences given in Example 3.1.6. Prove that each of these sequences converges and determine its limit.

3.12. Two important sequences with the same limit

In this section we study the sequences defined in (3.1.3) and (3.1.5).

$$E_n = \left(1 + \frac{1}{n}\right)^n, \quad n \in \mathbb{N},$$

$$S_1 = 2 \ \text{ and } \ \bigl(\forall\, n \in \mathbb{N}\bigr) \ \ S_{n+1} = S_n + \frac{1}{(n+1)!}.$$

Exercise 3.12.1. Prove by mathematical induction that $S_n \leq 3 - 1/n$ for all $n \in \mathbb{N}$.

Exercise 3.12.2. Prove that the sequence $\{S_n\}$ converges.

Exercise 3.12.3. Let $n, k \in \mathbb{N}$ and $n \geq k$. Use Bernoulli's inequality to prove that

$$\frac{n!}{(n-k)!\, n^k} \geq 1 - \frac{(k-1)k}{n}$$

HINT: Notice that

$$\frac{n!}{n^k (n-k)!} = 1 \cdot \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \geq \left(1 - \frac{k-1}{n}\right)^k.$$

Exercise 3.12.4. The following inequalities hold: $E_1 = S_1$ and for all integers n greater than 1,

$$S_n - \frac{3}{n} < E_n < S_n. \tag{3.12.1}$$

HINT: Let n be an integer greater than 2. Notice that by the Binomial Theorem

$$E_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{1}{n^k} = 1 + 1 + \sum_{k=2}^{n} \frac{n!}{(n-k)!\, n^k}\, \frac{1}{k!}.$$

Then use Exercise 3.12.3 to prove $E_n > S_n - S_{n-2}/n$. Then use Exercise 3.12.1.

Exercise 3.12.5. The sequences $\{E_n\}$ and $\{S_n\}$ converge to the same limit.

Exercise 3.12.5 justifies the following definition.

Definition 3.12.6. The number e is the common limit of the sequences $\{E_n\}$ and $\{S_n\}$.

Remark 3.12.7. The sequence $\{E_n\}$ is increasing. To prove this claim let $n \in \mathbb{N}$ be arbitrary. Consider the fraction

$$\begin{aligned} \frac{E_{n+1}}{E_n} &= \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n}} = \frac{n+1}{n}\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}} = \frac{n+1}{n}\left(\frac{\frac{n+2}{n+1}}{\frac{n+1}{n}}\right)^{n+1} \\ &= \frac{n+1}{n}\left(\frac{n(n+2)}{(n+1)^2}\right)^{n+1} = \frac{n+1}{n}\left(1-\frac{1}{(n+1)^2}\right)^{n+1} \end{aligned} \tag{3.12.2}$$

Since $-\frac{1}{(n+1)^2} > -1$ for all $n \in \mathbb{N}$, applying Bernoulli's Inequality with $x = -\frac{1}{(n+1)^2}$ we get

$$\left(1-\frac{1}{(n+1)^2}\right)^{n+1} > 1-(n+1)\frac{1}{(n+1)^2} = 1-\frac{1}{n+1}. \tag{3.12.3}$$

The relations (3.12.2) and (3.12.3) imply

$$\frac{E_{n+1}}{E_n} = \frac{n+1}{n}\left(1-\frac{1}{(n+1)^2}\right)^{n+1} > \frac{n+1}{n}\left(1-\frac{1}{n+1}\right) = 1.$$

Thus

$$\frac{E_{n+1}}{E_n} > 1 \quad \text{for all} \quad n \in \mathbb{N},$$

that is the sequence $\{E_n\}$ is increasing.

3.13. Subsequences

Composing functions is a common way how functions interact with each other. Can we compose two sequences? Let $x : \mathbb{N} \to \mathbb{R}$ and $y : \mathbb{N} \to \mathbb{R}$ be two sequences. Does the composition $x \circ y$ make sense? This composition makes sense only if the range of y is contained in $\mathbb{N}$. In this case $y : \mathbb{N} \to \mathbb{N}$. That is the composition $x \circ y$ makes sense only if y is a sequence in $\mathbb{N}$. It turns out that the most important composition of sequences involve increasing sequences in $\mathbb{N}$. In this section the Greek letters μ and ν will always denote increasing sequences of natural numbers.

Definition 3.13.1. A *subsequence* of a sequence $\{x_n\}$ is a composition of the sequence $\{x_n\}$ and an increasing sequence $\{\mu_k\}$ of natural numbers. This composition will be denoted by $\{x_{\mu_k}\}$ or $\{x(\mu_k)\}$.

Remark 3.13.2. The concept of subsequence consists of two ingredients:

- the sequence $\{x_n\}$ (remember it's really a function: $x : \mathbb{N} \to \mathbb{R}$)
- the increasing sequence $\{\mu_k\}$ of natural numbers (remember this is an increasing function: $\mu : \mathbb{N} \to \mathbb{N}$).

The composition $x \circ \mu$ of these two sequences is a new sequence $y : \mathbb{N} \to \mathbb{R}$. The k-th term y_k of this sequence is $y_k = x_{\mu_k}$. Note the analogy with the usual notation for functions: $y(k) = x(\mu(k))$. Usually we will not introduce the new name for a subsequence: we will write $\{x_{\mu_k}\}_{k=1}^{\infty}$ to denote a subsequence of the sequence $\{x_n\}$. Here $\{\mu_k\}_{k=1}^{\infty}$ is and increasing sequence of natural numbers which selects particular elements of the sequence $\{x_n\}$ to be included in the subsequence.

Remark 3.13.3. Roughly speaking, a subsequence of $\{x_n\}$ is a sequence formed by selecting some of the terms in $\{x_n\}$, keeping them in the same order as in the original sequence. It is the sequence $\{\mu_k\}$ of positive integers that does the selecting.

Example 3.13.4. Few examples of increasing sequences in $\mathbb{N}$ are:

(1) $\mu_k = 2k, \quad k \in \mathbb{N}$. (The sequence of even positive integers.)
(2) $\nu_k = 2k-1, \quad k \in \mathbb{N}$. (The sequence of odd positive integers.)
(3) $\mu_k = k^2, \quad k \in \mathbb{N}$. (The sequence of perfect squares.)
(4) Let j be a fixed positive integer. Set $\nu_k = j + k$ for all $k \in \mathbb{N}$.
(5) The sequence $2, 3, 5, 7, 11, 13, 17, \ldots$ of prime numbers. For this sequence no formula for $\{\mu_k\}$ is known.

Exercise 3.13.5. Let $\{\mu_n\}$ be an increasing sequence in $\mathbb{N}$. Prove that $\mu_n \geq n$ for all $n \in \mathbb{N}$.

Exercise 3.13.6. Each subsequence of a convergent sequence is convergent with the same limit.

Remark 3.13.7. The "contrapositive" of Exercise 3.13.6 is a powerful tool for proving that a given sequence does not converge. As an illustration prove that the sequence $\{(-1)^n\}$ does not converge in two different ways: using the definition of convergence and using the "contrapositive" of Exercise 3.13.6.

Exercise 3.13.8 (The Zipper Theorem)**.** Let $\{x_n\}$ be a sequence in $\mathbb{R}$ and let $\{\mu_k\}$ and $\{\nu_k\}$ be increasing sequences in $\mathbb{N}$. Assume

(a) $\{\mu_k : k \in \mathbb{N}\} \bigcup \{\nu_k : k \in \mathbb{N}\} = \mathbb{N}$.
(b) $\{x_{\mu_k}\}$ converges to L.
(c) $\{x_{\nu_k}\}$ converges to L.

Prove that $\{x_n\}$ converges to L.

Example 3.13.9. The sequence (o) in Example 3.1.4 does not converge, but it does have convergent subsequences, for instance the subsequence $\left\{\dfrac{2k}{2k+1}\right\}_{k=1}^{\infty}$ (Here $\mu_k = 2k,\ k \in \mathbb{N}$) and the subsequence $\left\{\dfrac{1}{(2k-1)2k}\right\}_{k=1}^{\infty}$ (Here $\nu_k = 2k-1,\ k \in \mathbb{N}$).

Remark 3.13.10. The notation for subsequences is a little tricky at first. Note that in x_{μ_k} it is k that is the variable. Thus the successive elements of the subsequence are $x_{\mu_1}, x_{\mu_2}, x_{\mu_3}$, etc. To indicate a different subsequence of the same sequence $\{x_n\}_{n=1}^{\infty}$ it would be necessary to change not the variable name, but the selection sequence. For example $\{x_{\mu_k}\}_{k=1}^{\infty}$ and $\{x_{\nu_k}\}_{k=1}^{\infty}$ in Example 3.13.9 are distinct subsequences of $\{x_n\}$. (Thus $\{x_{\mu_k}\}_{k=1}^{\infty}$ and $\{x_{\mu_j}\}_{j=1}^{\infty}$ are the same subsequence of $\{x_n\}_{n=1}^{\infty}$ for exactly the same reason that $x \mapsto x^2\ (x \in \mathbb{R})$ and $t \mapsto t^2\ (t \in \mathbb{R})$ are the same function. To make a different function it's the rule you must change, not the variable name.)

Example 3.13.11. Let $\{x_n\}$ be the sequence defined by

$$x_n = \frac{(-1)^n (n+1)^{(-1)^n}}{n}, \qquad n \in \mathbb{N}.$$

The values of $\{x_n\}$ are

$$-\frac{1}{1\cdot 2},\ \frac{3}{2},\ -\frac{1}{3\cdot 4},\ \frac{5}{4},\ -\frac{1}{5\cdot 6},\ \frac{7}{6},\ -\frac{1}{7\cdot 8},\ \frac{9}{8},\ -\frac{1}{9\cdot 10},\ \frac{11}{10},\ \ldots.$$

Exercise 3.13.12. Every sequence has a monotonic subsequence.

HINT: Let $\{x_n\}$ be an arbitrary sequence. Consider the set

$$\mathbb{M} = \{n \in \mathbb{N} : \forall\, k > n \text{ we have } x_k \geq x_n\}.$$

The set $\mathbb{M}$ is either finite or infinite. Construct a monotonic subsequence in each case.

Exercise 3.13.13. Every bounded sequence of real numbers has a convergent subsequence.

3.14. The Cauchy criterion

Definition 3.14.1. A sequence $\{s_n\}$ of real numbers is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists a real number N_ϵ such that

$$\forall\, n, m \in \mathbb{N}, \quad n, m > N_\epsilon \quad \Rightarrow \quad |s_n - s_m| < \epsilon.$$

Exercise 3.14.2. Prove that every convergent sequence is a Cauchy sequence.

Exercise 3.14.3. Prove that every Cauchy sequence is bounded.

Exercise 3.14.4. If a Cauchy sequence has a convergent subsequence, then it converges.

Exercise 3.14.5. Prove that each Cauchy sequence has a convergent subsequence.

Exercise 3.14.6. Prove that a sequence converges if and only if it is a Cauchy sequence.

3.15. Sequences and supremum and infimum

Exercise 3.15.1. Let $A \subset \mathbb{R}$, $A \neq \emptyset$ and assume that A is bounded above. Prove that $a = \sup A$ if and only if

(a) a is an upper bound of A, that is, $a \geq x$, for all $x \in A$;
(b) there exists a sequence $\{x_n\}$ such that

$$x_n \in A \quad \text{for each} \quad n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to\infty} x_n = a.$$

Exercise 3.15.2. Let $A \subset \mathbb{R}$, $A \neq \emptyset$ and assume that A is bounded above. Let $a = \sup A$ and assume that $a \notin A$. Prove that there exists a strictly increasing sequence $\{x_n\}$ such that

$$x_n \in A \quad \text{for each} \quad n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to\infty} x_n = a.$$

Exercise 3.15.3. State and prove the characterization of infimum which is analogous to the characterization of $\sup A$ given in Exercise 3.15.1.

Exercise 3.15.4. State and prove an exercise involving infimum of a set which is analogous to Exercise 3.15.2.

CHAPTER 4

Continuous functions

In this chapter I will always denote a non-empty subset of $\mathbb{R}$. This includes more general sets, but the most common examples of I are intervals.

4.1. The ϵ-δ definition of a continuous function

Definition 4.1.1. A function $f : I \to \mathbb{R}$ is *continuous at a point* $x_0 \in I$ if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon, x_0) > 0$ such that

$$x \in \bigl(x_0 - \delta, x_0 + \delta\bigr) \cap I \qquad \Rightarrow \qquad |f(x) - f(x_0)| < \epsilon. \tag{4.1.1}$$

The function f is *continuous on* I if it is continuous at each point of I.

Note that the implication in (4.1.1) can be restated as

$$x \in I \ \text{ and } \ |x - x_0| < \delta(\epsilon, x_0) \qquad \Rightarrow \qquad |f(x) - f(x_0)| < \epsilon.$$

Next we restate Definition 4.1.1 using the terminology introduced in Section 2.14. For a function $f : I \to \mathbb{R}$ and a subset $A \subseteq I$ we will use the notation $f(A)$ to denote the set $\bigl\{y \in \mathbb{R} : \exists\, x \in A \text{ s.t. } f(x) = y\bigr\} = \bigl\{f(x) : x \in A\bigr\}$.

A function $f : I \to \mathbb{R}$ is continuous at a point $x_0 \in I$ if for each neighborhood V of $f(x_0)$ there exists a neighborhood U of x_0 such that

$$f\bigl(I \cap U\bigr) \subseteq V.$$

4.2. Finding $\delta(\epsilon)$ for a given function at a given point

In this and the next section we will prove that some familiar functions are continuous. This should be a review of what was done in Math 226.

A general strategy for proving that a given function f is continuous at a given point x_0 is as follows:

Step 1. Simplify the expression $|f(x) - f(x_0)|$ and try to establish a simple connection with the expression $|x - x_0|$. The simplest connection is to discover positive constants δ_0 and K such that

$$x \in I \ \text{ and } \ x_0 - \delta_0 < x < x_0 + \delta_0 \qquad \Rightarrow \qquad |f(x) - f(x_0)| \leq K\,|x - x_0|. \tag{4.2.1}$$

Constants δ_0 and K might depend on x_0. Formulate your discovery as a lemma.

Step 2. Let $\epsilon > 0$ be given. Use the result in Step 1 to define your $\delta(\epsilon, x_0)$. For example, if (4.2.1) holds, then $\delta(\epsilon, x_0) = \min\bigl\{\epsilon/K, \delta_0\bigr\}$.

Step 3. Use the definition of $\delta(\epsilon, x_0)$ from Step 2 and the lemma from Step 1 to prove the implication (4.1.1).

Example 4.2.1. We will show that the function $f(x) = x^2$ is continuous at $x_0 = 3$. Here $I = \mathbb{R}$ and we do not need to worry about the domain of f.
Step 1. First simplify

$$\big|f(x) - f(x_0)\big| = \big|x^2 - 3^2\big| = \big|(x+3)(x-3)\big| = \big|x+3\big|\,\big|x-3\big|. \tag{4.2.2}$$

Now we notice that if $2 < x < 4$ we have $\big|x+3\big| = x+3 \leq 7$. Thus (4.2.1) holds with $\delta_0 = 1$ and $K = 7$. We formulate this result as a lemma.

Lemma. *Let* $f(x) = x^2$ *and* $x_0 = 3$. *Then*

$$|x-3| < 1 \quad \Rightarrow \quad \big|x^2 - 3^2\big| < 7|x-3|. \tag{4.2.3}$$

Proof. Let $|x-3| < 1$. Then $2 < x < 4$. Therefore $x+3 > 0$ and $|x+3| = x+3 < 7$. By (4.2.2) we now have $\big|x^2 - 3^2\big| < 7|x-3|$. □

Step 2. Now we define $\delta(\epsilon) = \min\big\{\epsilon/7, 1\big\}$.
Step 3. It remains to prove (4.1.1). To this end, assume $|x-3| < \min\big\{\epsilon/7, 1\big\}$. Then $|x-3| < 1$. Therefore, by Lemma we have $\big|x^2 - 3^2\big| < 7|x-3|$. Since by the assumption $|x-3| < \epsilon/7$, we have $7|x-3| < 7\,\epsilon/7\epsilon$. Now the inequalities

$$\big|x^2 - 3^2\big| < 7|x-3| \quad \text{and} \quad 7|x-3| < \epsilon$$

imply that $\big|x^2 - 3^2\big| < \epsilon$. This proves (4.1.1) and completes the proof that the function $f(x) = x^2$ is continuous at $x_0 = 3$.

Exercise 4.2.2. Prove that the reciprocal function $x \mapsto \dfrac{1}{x}$, $x \neq 0$, is continuous at $x_0 = 1/2$.

Exercise 4.2.3. State carefully what it means for a function f *not* to be continuous at a point x_0 in its domain. (Express this as a formal mathematical statement.)

Exercise 4.2.4. Consider the function $f(x) = \operatorname{sgn} x$. Find a point x_0 at which the function f is not continuous. Provide a formal proof.

Exercise 4.2.5. Show that the function $f(x) = x^2$ is continuous on $\mathbb{R}$.

Exercise 4.2.6. Prove that $q(x) = 3x^2 + 5$ is continuous on $\mathbb{R}$.

4.3. Familiar continuous functions

Exercise 4.3.1. Let $m, k \in \mathbb{R}$ and $m \neq 0$. Prove that the linear function $\ell(x) = m\,x + k$ is continuous on $\mathbb{R}$.

Exercise 4.3.2. Let $a, b, c \in \mathbb{R}$ and $a \neq 0$. Prove that the quadratic function $q(x) = a\,x^2 + b\,x + c$ is continuous on $\mathbb{R}$.

Exercise 4.3.3. Let $n \in \mathbb{N}$ and let $x, x_0 \in \mathbb{R}$ be such that $x_0 - 1 \leq x \leq x_0 + 1$. Prove the following inequality

$$\big|x^n - x_0^n\big| \leq n\big(|x_0| + 1\big)^{n-1}\big|x - x_0\big|.$$

Hint: First notice that the assumption $x_0 - 1 \leq x \leq x_0 + 1$ implies that $|x| < |x_0| + 1$. Then use the Mathematical Induction and the identity

$$\big|x^{n+1} - x_0^{n+1}\big| = \big|x^{n+1} - x\,x_0^n + x\,x_0^n - x_0^{n+1}\big|.$$

Exercise 4.3.4. Let $n \in \mathbb{N}$. Prove that the power function $x \mapsto x^n$, $x \in \mathbb{R}$, is continuous on $\mathbb{R}$.

Exercise 4.3.5. Let $n \in \mathbb{N}$ and let $a_0, a_1, \dots, a_n \in \mathbb{R}$ with $a_n \neq 0$. Prove that the n-th order polynomial

$$p(x) = a_0 + a_1\, x + \cdots + a_{n-1}\, x^{n-1} + a_n\, x^n$$

is a continuous function on $\mathbb{R}$.

Exercise 4.3.6. Prove that the reciprocal function $x \mapsto \dfrac{1}{x}$, $x \neq 0$, is continuous on its domain.

Exercise 4.3.7. Prove that the square root function $x \mapsto \sqrt{x}$, $x \geq 0$, is continuous on its domain.

Exercise 4.3.8. Let $n \in \mathbb{N}$ and let x and a be positive real numbers. Prove that

$$\bigl| \sqrt[n]{x} - \sqrt[n]{a} \bigr| \leq \frac{\sqrt[n]{a}}{a}\, \bigl| x - a \bigr|.$$

HINT: Notice that the given inequality is equivalent to

$$b^{n-1}\, \bigl| y - b \bigr| \leq \bigl| y^n - b^n \bigr|, \quad y, b > 0.$$

This inequality can be proved using Exercise 2.7.7 (with $a = 1$ and $x = y/b$).

Exercise 4.3.9. Let $n \in \mathbb{N}$. Prove that the n-th root function $x \mapsto \sqrt[n]{x}$, $x \geq 0$, is continuous on its domain.

4.4. Various properties of continuous functions

Exercise 4.4.1. Let $f : I \to \mathbb{R}$ be continuous at $x_0 \in I$ and let y be a real number such that $f(x_0) < y$. Then there exists $\alpha > 0$ such that

$$x \in I \cap (x_0 - \alpha, x_0 + \alpha) \qquad \Rightarrow \qquad f(x) < y.$$

Illustrate with a diagram.

Exercise 4.4.2. Let $f : I \to \mathbb{R}$ be a continuous function on I. Let S be a nonempty bounded above subset of I such that $u = \sup S$ belongs to I. Let $y \in \mathbb{R}$. Prove: If $f(x) \leq y$ for each $x \in S$, then $f(u) \leq y$.

The following exercise establishes a connection between continuous functions and convergent sequences.

It is very important since it will enable us to use what we learned about sequences to study continuous functions.

Exercise 4.4.3. Let $f : I \to \mathbb{R}$ be continuous at $x_0 \in I$. Let $\{t_n\}$ be a sequence in I that converges to $x_0 \in I$. Then $f\,(t_n) \to f(x_0)$ as $n \to \infty$.

Exercise 4.4.4. Let $f : I \to \mathbb{R}$ be continuous at $x_0 \in I$. Let $\{t_n\}$ be a sequence in I that converges to $x_0 \in I$. Assume that there is a real number y such that $f(t_n) \leq y$ for all $n \in \mathbb{N}$. Then $f(x_0) \leq y$.

Exercise 4.4.5. Let $f : I \to \mathbb{R}$ be continuous at $x_0 \in I$. Let $\{x_n\}$ be a sequence in I that converges to $x_0 \in I$. Assume that there is a real number y such that $f(t_n) \geq y$ for all $n \in \mathbb{N}$. Then $f(x_0) \geq y$.

4.5. Algebra of continuous functions

All exercises in this section have the same structure. With the exception of Exercise 4.5.3, there are three functions in each exercise: f, g and h. The function h is always related in a simple (green) way to the functions f and g. Based on the given (green) information about f and g you are asked to prove a claim (red) about the function h.

Exercise 4.5.1. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be given functions with a common domain. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = f(x) + g(x), \qquad x \in I.$$

(a) If f and g are continuous at $x_0 \in I$, then h is continuous at x_0.
(b) If f and g are continuous on I, then h is continuous on I.

Exercise 4.5.2. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be given functions with a common domain. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = f(x)g(x), \qquad x \in I.$$

(a) If f and g are continuous at $x_0 \in I$, then h is continuous at x_0.
(b) If f and g are continuous on I, then h is continuous on I.

Exercise 4.5.3. Let $g : I \to \mathbb{R}$ be a given functions such that $g(x) \neq 0$ for all $x \in I$. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = \frac{1}{g(x)}, \qquad x \in I.$$

(a) If g is continuous at $x_0 \in I$, then h is continuous at x_0.
(b) If g is continuous on I, then h is continuous on I.

Exercise 4.5.4. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be given functions with a common domain. Assume that $g(x) \neq 0$ for all $x \in I$. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = \frac{f(x)}{g(x)}, \qquad x \in I.$$

(a) If f and g are continuous at $x_0 \in I$, then h is continuous at x_0.
(b) If f and g are continuous on I, then h is continuous on I.

Exercise 4.5.5. Let I and J be non-empty subsets of $\mathbb{R}$. Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be given functions. Assume that the range of f is contained in J. Define the function $h : I \to \mathbb{R}$ by

$$h(x) = g(f(x)), \qquad x \in I.$$

(a) If f is continuous at $x_0 \in I$ and g is continuous at $f(x_0) \in J$, then h is continuous at x_0.
(b) If f is continuous on I and g is continuous on J, then h is continuous on I.

4.6. Continuous functions on a closed bounded interval $[a, b]$

In this section we assume that $a, b \in \mathbb{R}$ and $a < b$.

Exercise 4.6.1. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. If $f(a) < 0$ and $f(b) > 0$, then there exists $c \in [a,b]$ such that $f(c) = 0$.

HINT: Consider the set

$$W = \bigl\{ w \in [a,b) : \forall x \in [a,w]\ f(x) < 0 \bigr\}.$$

Prove the following properties of W:

(i) W does not have a maximum.
(ii) W has a supremum. Set $w = \sup W$.
(iii) Review Exercise 4.4.2.
(iv) Connect the dots.

Exercise 4.6.2. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then there exists $c \in [a,b]$ such that $f(x) \leq f(c)$ for all $x \in [a,b]$.

HINT: Consider the set

$$W = \Bigl\{ v \in [a,b) : \exists\ z \in [a,b] \ \text{ such that } \ \forall\ x \in [a,v]\ \ f(x) < f(z) \Bigr\}.$$

Here $[a,a]$ denotes the set $\{a\}$. Prove the following properties of the set W:

(i) If $a < u$ and $[a,u) \subseteq W$ and there exists $t \in [a,b]$ such that $f(t) > f(u)$, then $u \in W$.
(ii) W does not have a maximum.
(iii) W has a supremum. Set $w = \sup W$ and prove $[a,w) \subseteq W$.
(iv) The items (ii) and (iii) yield information about w.

Exercise 4.6.3. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then there exists $d \in [a,b]$ such that $f(d) \leq f(x)$ for all $x \in [a,b]$.

HINT: Use Exercise 4.6.2.

Exercise 4.6.4. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then the range of f is a closed bounded interval.

HINT: Use Exercises 4.6.2, 4.6.3, and 4.6.1.

Exercise 4.6.5. Consider the function $f(x) = x^2$, $x \in \mathbb{R}$.

(a) Prove that 2 is in the range of f.
(b) Prove that the range of f equals $[0, +\infty)$.

Definition 4.6.6. A function f is *increasing* on an interval I if $x, y \in I$ and $x < y$ imply $f(x) < f(y)$. A function f is *decreasing* if $x, y \in I$ and $x < y$ imply $f(x) > f(y)$. A function which is increasing or decreasing is said to be *strictly monotonic*.

Exercise 4.6.7. If f is continuous and increasing on $[a,b]$ or continuous and decreasing on $[a,b]$, then for each y between $f(a)$ and $f(b)$ there is exactly one $x \in [a,b]$ such that $f(x) = y$.

Exercise 4.6.8. Let $f(x) = x^3 + x$, $x \in \mathbb{R}$. Prove that f has an inverse. That is, prove that for each $y \in \mathbb{R}$ there exists unique $x \in \mathbb{R}$ such that $f(x) = y$.