8. The initial value problem to be solved is

$$y''(t) + 144y(t) = \cos(11t), \quad y(0) = y_0, \quad y'(0) = 0.$$

The solution of this IVP is

$$y(t) = \frac{1}{23}\big(\cos(11t) - \cos(12t)\big) + y_0\cos(12t) = \frac{2}{23}\sin\left(\frac{1}{2}t\right)\sin\left(\frac{23}{2}t\right) + y_0\cos(12t)$$

Figures 1 through 6 show the solutions for the indicated initial values. We notice that the “fast” oscillations are still present while the “slow” oscillations are less and less pronounced as y_0 increases.

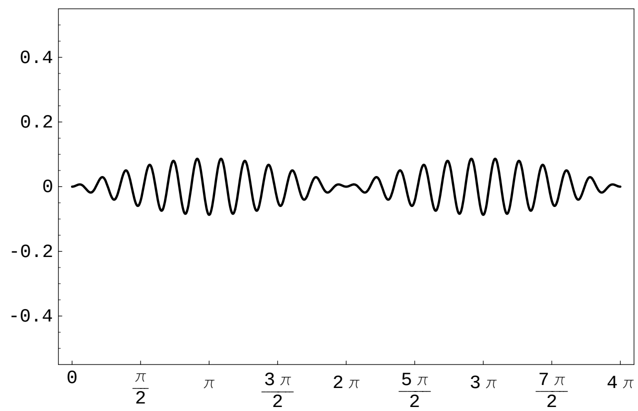

Figure 1: $y_0 = 0$

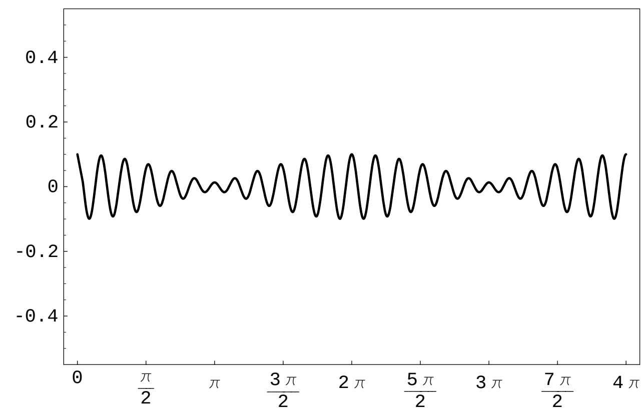

Figure 2: $y_0 = 0.1$

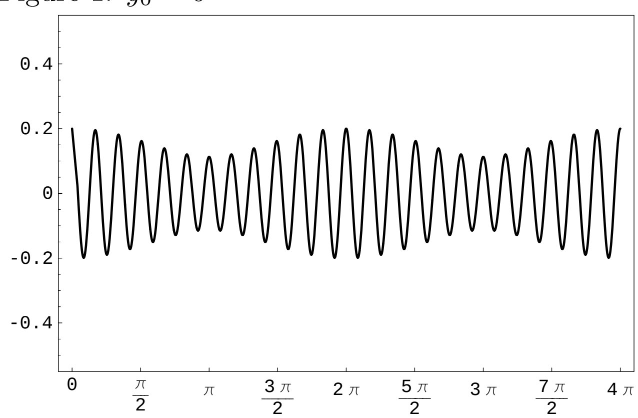

Figure 3: $y_0 = 0.2$

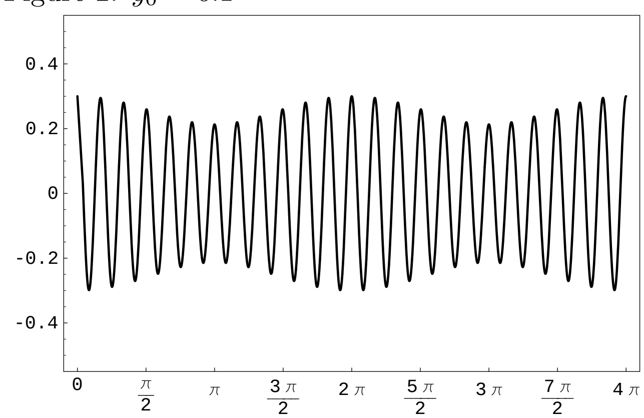

Figure 4: $y_0 = 0.3$

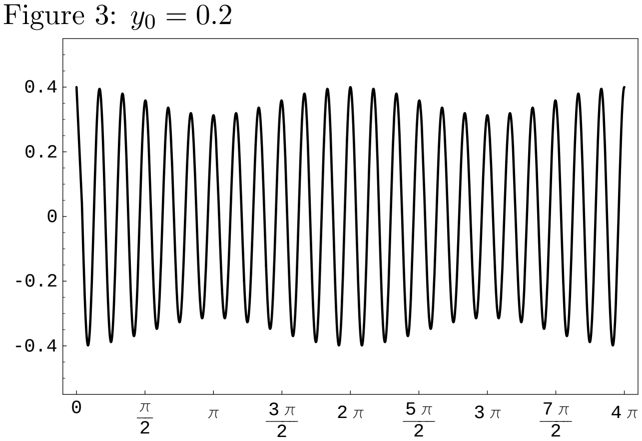

Figure 5: $y_0 = 0.4$

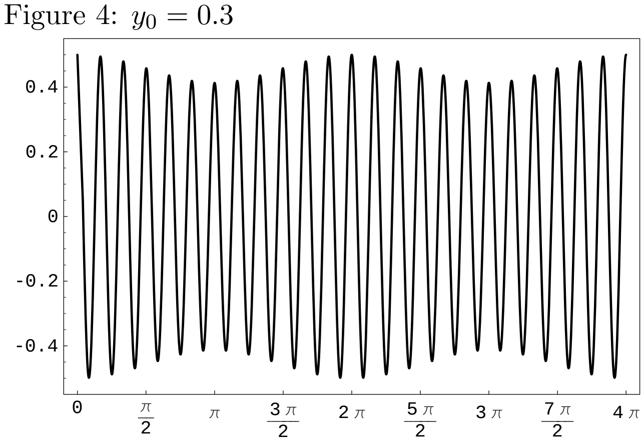

Figure 6: $y_0 = 0.5$

9. The initial value problem to be solved is

$$x''(t) + 4x(t) = 4\cos(\omega t), \quad x(0) = 0, \quad x'(0) = 0.$$

Here we assume that $\omega \neq 2$. The solution of this initial value problem is

$$x(t) = \frac{4}{\omega^2 - 4}\big(\cos(2t) - \cos(\omega t)\big)$$

This function written in the "beats" form is

$$x(t) = \frac{8}{\omega^2 - 4}\sin\left(\frac{\omega - 2}{2}t\right)\sin\left(\frac{\omega + 2}{2}t\right)$$

From this formula we see that the envelope of the beating motion is the determined by

$$\frac{8}{\omega^2 - 4}\sin\left(\frac{\omega - 2}{2}\right), \qquad \text{and} \qquad -\frac{8}{\omega^2 - 4}\sin\left(\frac{\omega - 2}{2}\right)$$

In Figures 7 and 8 the envelope is pictured in blue.

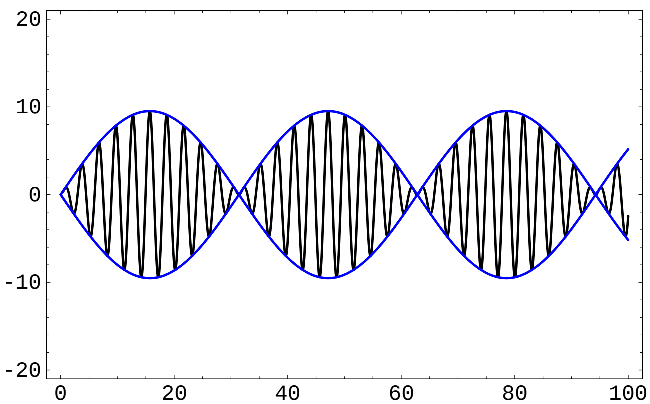

Figure 7: $\omega = 2.2$

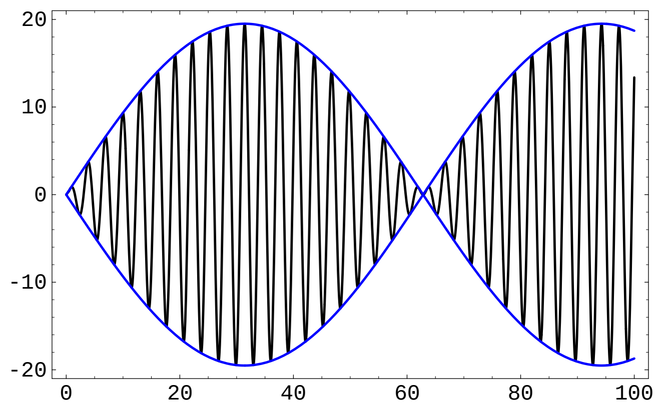

Figure 8: $\omega = 2.1$

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10. The initial value problem to be solved is

$$x''(t) + 25x(t) = 4\cos(5t), \quad x(0) = 1, \quad x'(0) = 0.$$

The general solution of the corresponding homogeneous equation is

$$x_h(t) = C_1\cos(5t) + C_2\sin(5t), \quad \text{or, in complex form,} \quad x_h(t) = C_1 e^{-5it} + C_2 e^{-5it}.$$

The first guess for the particular solution would be ae^{5it}, but since this function is a part of x_h, we should try

$$\begin{aligned} z_p(t) &= ate^{5it}, \\ z_p'(t) &= ae^{5it} + ia5te^{5it} = ae^{5it}\big(1 + 5it\big) \\ z_p''(t) &= a5ie^{5it}\big(1 + 5it\big) + a5ie^{5it} = ae^{5it}\big(10i - 25t\big) \end{aligned}$$

Substituting in the given equation we get

$$ae^{5it}\big(10i - 25t\big) + 25ate^{5it} = 4e^{5it},$$

which simplifies to

$$10ai = 4, \qquad a = -\frac{2}{5}\,i$$

Hence the complex particular solution is

$$z_p(t) = -\frac{2}{5} i t e^{5it}$$

But we are interested in the real part only. Hence

$$x_p(t) = \frac{2}{5}\, t \sin(5t)$$

The general solution of the given equation is

$$x(t) = x_h(t) + x_p(t) = C_1 \cos(5t) + C_2 \sin(5t) + \frac{2}{5}\, t \sin(5t)$$

To find the solution of the initial value problem we need

$$x'(t) = -5C_1 \sin(5t) + 5C_2 \cos(5t) + \frac{2}{5} \sin(5t) + 2\, t \cos(5t)$$

Hence

$$1 = C_1, \quad 0 = 5C_2.$$

Thus the solution of the initial value problem is

$$x(t) = \cos(5t) + \frac{2}{5}\, t \sin(5t)$$

The function $\frac{2}{5}\, t \sin(5t)$ oscillates between the lines $\frac{2}{5}\, t$ and $-\frac{2}{5}\, t$. This constitutes the resonance, since the amplitude grows without bound.

---

12. The characteristic polynomial of the given differential equation is

$$P(\lambda) = \lambda^2 + \lambda + 4.$$

We calculate

$$P(2i) = -4 + 2i + 4 = 2i = 2 \exp^{i\frac{\pi}{2}}.$$

Hence, the transfer function is

$$H(2i) = \frac{1}{P(2i)} = \frac{1}{2} \exp^{-i\frac{\pi}{2}}$$

Therefore the complex particular solution is

$$z_p(t) = \frac{1}{2} \exp^{-i\frac{\pi}{2}}\ 3 \exp^{2it} = \frac{3}{2}\ \exp^{(2t-\frac{\pi}{2})i}$$

We are interested only in the real part

$$x_p(t) = \frac{3}{2} \cos\Big(2t - \frac{\pi}{2}\Big) = \frac{3}{2} \sin(2t).$$

This is the steady state solution.

---

14. The characteristic polynomial of the given differential equation is

$$P(\lambda) = \lambda^2 + 2\lambda + 4.$$

We calculate

$$P(2\pi i) = -4\pi^2 + 4\pi i + 4 = 4\big((1-\pi^2) + \pi i\big) = 4\sqrt{(1-\pi^2)^2 + \pi^2}\ \exp^{i\phi}.$$

Here $\phi = \arg\big((1-\pi^2)+\pi i\big)$. Since $1-\pi^2<0$ we can not use the function arctan directly, but we can use arccos, or we have to add π to arctan:

$$\phi = \arccos\left(\frac{1-\pi^2}{\sqrt{(1-\pi^2)^2+\pi^2}}\right) = \pi + \arctan\left(\frac{\pi}{1-\pi^2}\right) \approx 2.8012$$

Hence, the transfer function is

$$H(2\pi i) = \frac{1}{P(2\pi i)} = \frac{1}{4\sqrt{(1-\pi^2)^2+\pi^2}}\ \exp^{-i\phi}$$

Therefore the complex particular solution is

$$z_p(t) = \frac{1}{4\sqrt{(1-\pi^2)^2+\pi^2}}\ \exp^{-i\phi}\ 2\ \exp^{2\pi i t} = \frac{1}{2\sqrt{(1-\pi^2)^2+\pi^2}}\ \exp^{(2\pi t-\phi)i}$$

We are interested only in the **imaginary part**

$$x_p(t) = \frac{1}{2\sqrt{(1-\pi^2)^2+\pi^2}}\ \sin(2\pi t-\phi)\,.$$

This is the steady state solution.

---

16. The initial value problem to be solved is

$$x''(t)+5x'(t)+4x(t) = 2\sin(2t), \quad x(0)=1, \quad x'(0)=0.$$

The general solution of the corresponding homogeneous equation is

$$x_h(t) = C_1e^{-4t}+C_2e^{-t}.$$

The guess for the particular solution is ae^{2it}. This guess leads to the equation involving the characteristic polynomial of the given differential equation:

$$P(\lambda) = \lambda^2+5\lambda+4,$$

that is $aP(2i)=2$. Since

$$P(2i) = -4+10i+4 = 10i = 10e^{i\frac{\pi}{2}}.$$

Hence the complex particular solution is

$$z_p(t) = \frac{1}{5}\,e^{-i\frac{\pi}{2}}\,e^{2it} = \frac{1}{5}\,e^{(2t-\frac{\pi}{2})i}$$

But we are interested in the imaginary part only. Hence

$$x_p(t) = \frac{1}{5}\,\sin\left(2t-\frac{\pi}{2}\right) = -\frac{1}{5}\,\cos(2t)$$

The general solution of the given equation is

$$x(t) = x_h(t)+x_p(t) = C_1e^{-4t}+C_2e^{-t}-\frac{1}{5}\,\cos(2t)\,.$$

From the initial conditions we find the particular solution of the IVP:

$$x(t) = -\frac{2}{5}e^{-4t}+\frac{8}{5}e^{-t}-\frac{1}{5}\,\cos(2t)\,.$$

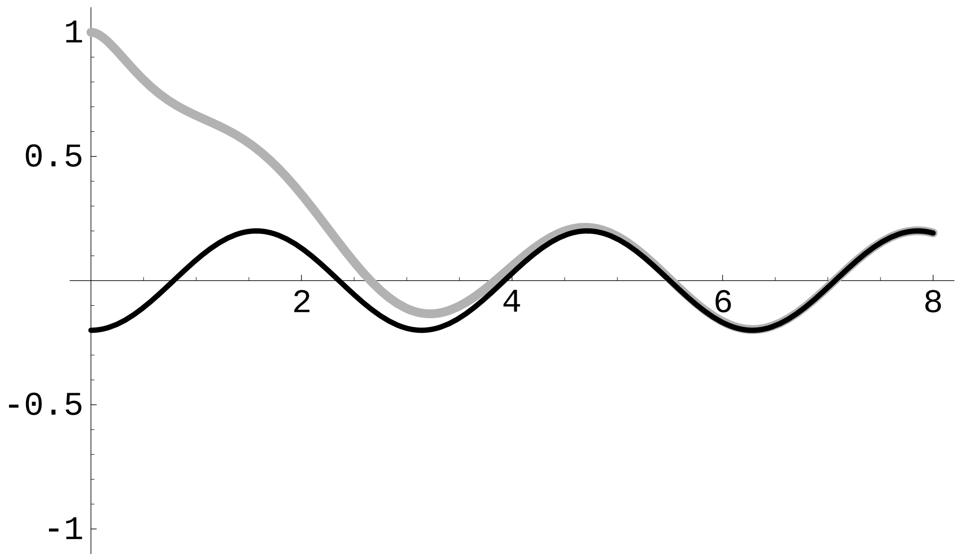

Figure 9: Problem 16

The transient term of the solution is

$$-\frac{2}{5}e^{-4t} + \frac{8}{5}e^{-t}$$

and the steady state solution is

$$-\frac{1}{5}\cos(2t).$$

Figure 9 shows the solution of the IVP and the steady state solution.

---

16 and 22. The initial value problem to be solved is

$$x''(t) + 2x'(t) + 2x(t) = \cos(2t), \quad x(0) = 0, \quad x'(0) = 2.$$

The general solution of the corresponding homogeneous equation is

$$x_h(t) = C_1 e^{-t}\cos t + C_2 e^{-t}\sin t.$$

The guess for the particular solution is ae^{2it}. This guess leads to the equation involving the characteristic polynomial of the given differential equation:

$$P(\lambda) = \lambda^2 + 2\lambda + 2,$$

that is $aP(2i) = 1$. Since

$$P(2i) = -4 + 4i + 2 = -2 + 4i = 2\sqrt{5}e^{i\phi}, \quad \text{where} \quad \phi = \arg(-1+2i) = \arccos\left(\frac{-1}{\sqrt{5}}\right) \approx 2.0344$$

Hence the complex particular solution is

$$z_p(t) = \frac{1}{2\sqrt{5}}\, e^{-i\phi}\, e^{2it} = \frac{1}{2\sqrt{5}}\, e^{(2t-\phi)i}$$

But we are interested in the real part only. Hence

$$x_p(t) = \frac{1}{2\sqrt{5}}\cos(2t - \phi)$$

The general solution of the given equation is

$$x(t) = x_h(t) + x_p(t) = C_1 e^{-t}\cos t + C_2 e^{-t}\sin t + \frac{1}{2\sqrt{5}}\cos(2t - \phi).$$

From the initial conditions we find the particular solution of the IVP:

$$x(t) = \frac{1}{10}e^{-t}\cos t + \frac{17}{10}e^{-t}\sin t + \frac{1}{2\sqrt{5}}\cos(2t - \phi).$$

The transient term of the solution is

$$\frac{1}{10}e^{-t}\cos t + \frac{17}{10}e^{-t}\sin t = \frac{\sqrt{1+17^2}}{10}\, e^{-t}\cos\big(t - \arctan(17)\big)$$

and the steady state solution is

$$\frac{1}{2\sqrt{5}}\cos(2t - \phi).$$

The time constant $T_C = 1$. It is important number here is

$$\frac{\sqrt{1+17^2}}{10}\, e^{-4} \approx 0.03119.$$

Figure 10 shows the transient term and the values -0.03119 and 0.03119. It is clear that for $t > 4 = 4T_C$ the absolute value of the transient term is < 0.03119.
Figure 11 shows the solution of the IVP and the steady state solution.

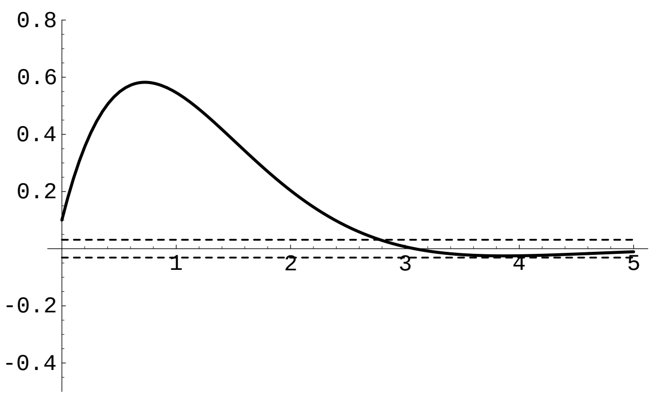

Figure 10: Problem 22, the transient term

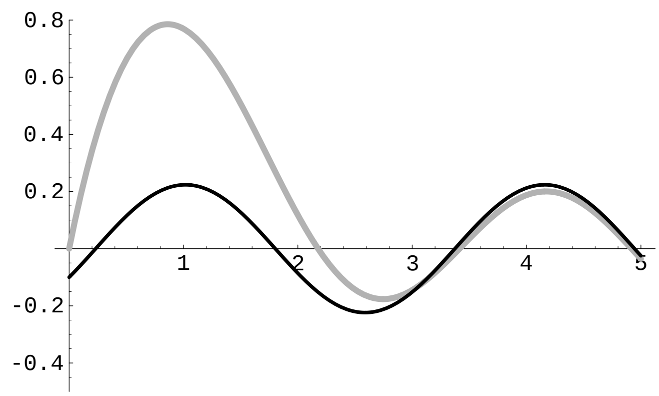

Figure 11: Pr.18, the IVP sol. and the st. st. sol.

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24. The equation to be solved is

$$x''(t) + 2x'(t) + 4x(t) = 3\cos(2t).$$

Here are the solutions of the given initial value problems:

$$x(t) = 2e^{-t}\cos(\sqrt{3}t) + \frac{\sqrt{3}}{6}e^{-t}\sin(\sqrt{3}t) + \frac{3}{4}\sin(2t) \quad \text{solves} \quad x(0) = 2,\ x'(0) = 0,$$

$$x(t) = e^{-t}\cos(\sqrt{3}t) - \frac{\sqrt{3}}{6}e^{-t}\sin(\sqrt{3}t) + \frac{3}{4}\sin(2t) \quad \text{solves} \quad x(0) = 1,\ x'(0) = 0,$$

$$x(t) = -\frac{\sqrt{3}}{2}e^{-t}\sin(\sqrt{3}t) + \frac{3}{4}\sin(2t) \quad \text{solves} \quad x(0) = 0,\ x'(0) = 0,$$

$$x(t) = -e^{-t}\cos(\sqrt{3}t) - \frac{5\sqrt{3}}{6}e^{-t}\sin(\sqrt{3}t) + \frac{3}{4}\sin(2t) \quad \text{solves} \quad x(0) = -1,\ x'(0) = 0,$$

$$x(t) = -2e^{-t}\cos(\sqrt{3}t) - \frac{7\sqrt{3}}{6}e^{-t}\sin(\sqrt{3}t) + \frac{3}{4}\sin(2t) \quad \text{solves} \quad x(0) = -2,\ x'(0) = 0,$$

Clearly the steady state solution is $\frac{3}{4}\sin(2t)$. Figure 12 shows the five solutions of the initial value problems together with the steady state solutions. It clearly shows that all these solution approach the steady state solution.

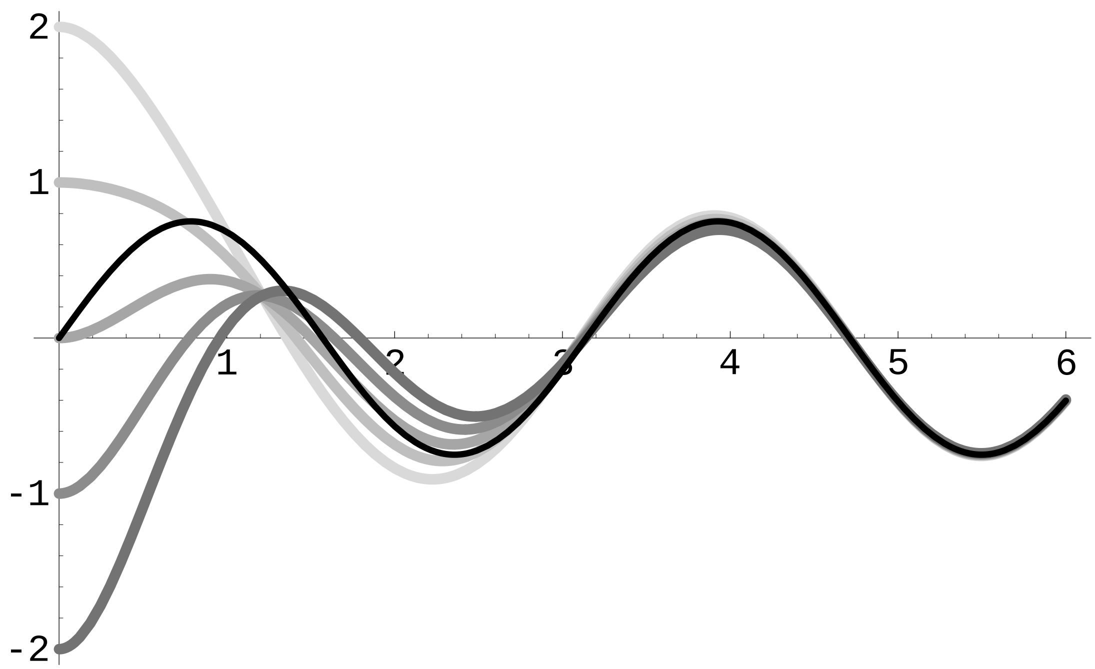

Figure 12: Problem 24

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26. The characteristic polynomial of the given differential equation is

$$P(\lambda) = \lambda^2 + \frac{1}{5}\lambda + 1.$$

To determine the transfer function and the gain we calculate

$$P(i\omega) = -\omega^2 + \frac{1}{5}i\omega + 1 = 1 - \omega^2 + \frac{1}{5}\omega i = \frac{1}{5}\sqrt{25(1-\omega^2)^2+1}e^{i\phi}, \quad \text{where} \quad \phi = \arccos\left(\frac{1}{\sqrt{25(1-\omega^2)^2+1}}\right).$$

Thus the transfer function, the gain and the phase are as follows:

$$H(i\omega) = \frac{5}{\sqrt{25(1-\omega^2)^2+1}}e^{-i\phi}, \quad R(\omega) = \frac{5}{\sqrt{25(1-\omega^2)^2+1}}, \quad \phi(\omega) = \arccos\left(\frac{1}{\sqrt{25(1-\omega^2)^2+1}}\right).$$

The particular values for $\omega = 1$ are

$$H(i) = 5e^{-i\frac{\pi}{2}}, \quad R(1) = 5, \quad \phi = \frac{\pi}{2}.$$

The steady state response is

$$\mathrm{Re}\left(5e^{-i\frac{\pi}{2}}e^{it}\right) = \mathrm{Re}\left(5\,e^{(t-\frac{\pi}{2})i}\right) = 5\cos\left(t - \frac{\pi}{2}\right) = 5\sin(t).$$

Figure 13 shows the driving function $\cos t$ in black and the steady state response in gray. The gain can be seen as a ratio of the amplitude of the steady state response (which is 5) and the amplitude of the driving function (which is 1). The phase can be seen as a smaller distance between consecutive zeros of the state response (for example $t = \pi$) and the driving function (for example $t = 3\pi/2$). Thus the phase is $\pi/2$.

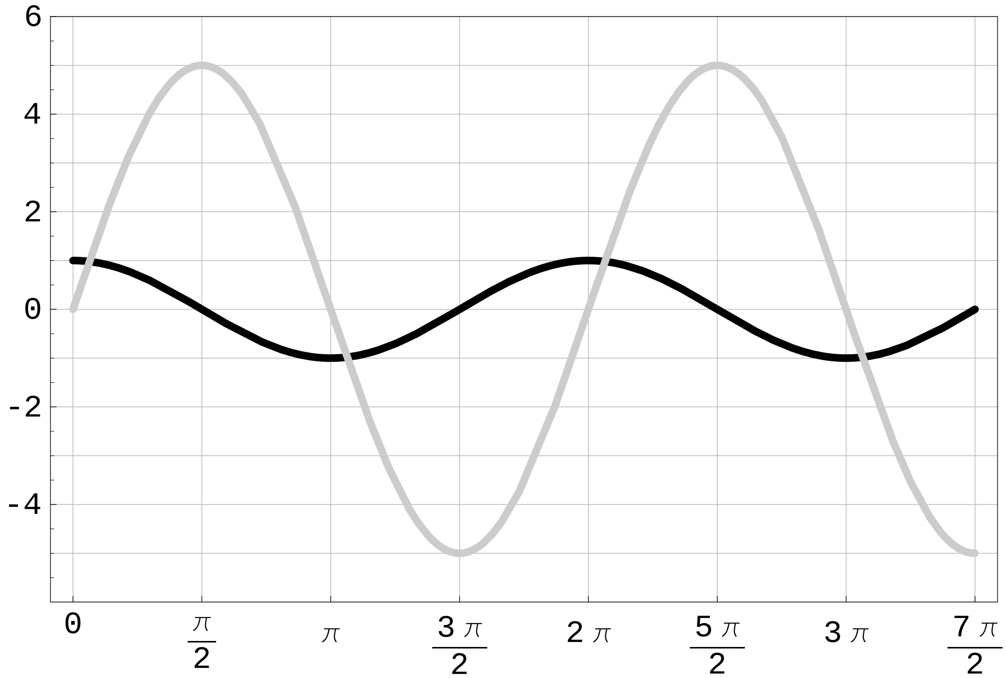

Figure 13: Problem 26

30. I estimate that the first zero of $\cos t$ in Figure 6 in the book is

$$8\pi + \frac{\pi}{2} \approx 26.7035.$$

The closest zero of the steady state response $x(t)$ is

$$8\pi + \frac{5\pi}{6} \approx 27.7507.$$

Therefore I estimate that the phase is given

$$\phi = \frac{\pi}{3} \approx 1.0472.$$

For simplicity I estimate that the amplitude of the steady state response $x(t)$ is 2.25. Thus, the gain is $R = 2.25 = 9/4$. Figure 14 is a reproduction of Figure 6 in the book.
The easiest way to calculate c and ω_0 in the equation

$$x''(t) + 2cx'(t) + \omega_0^2 x(t) = \cos(t).$$

is from the equation

$$P(i) = -1 + 2ci + \omega_0^2 = \omega_0^2 - 1 + 2ci = \frac{1}{R}\, e^{i\phi} = \frac{4}{9}\, e^{i\frac{\pi}{3}}.$$

Hence

$$\omega_0^2 - 1 + 2ci = \frac{4}{9}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right).$$

Therefore,

$$\omega_0^2 - 1 = \frac{2}{9}, \qquad 2c = \frac{2\sqrt{3}}{9},$$

that is,

$$\omega_0 = \frac{\sqrt{11}}{3}, \qquad c = \frac{\sqrt{3}}{9},$$

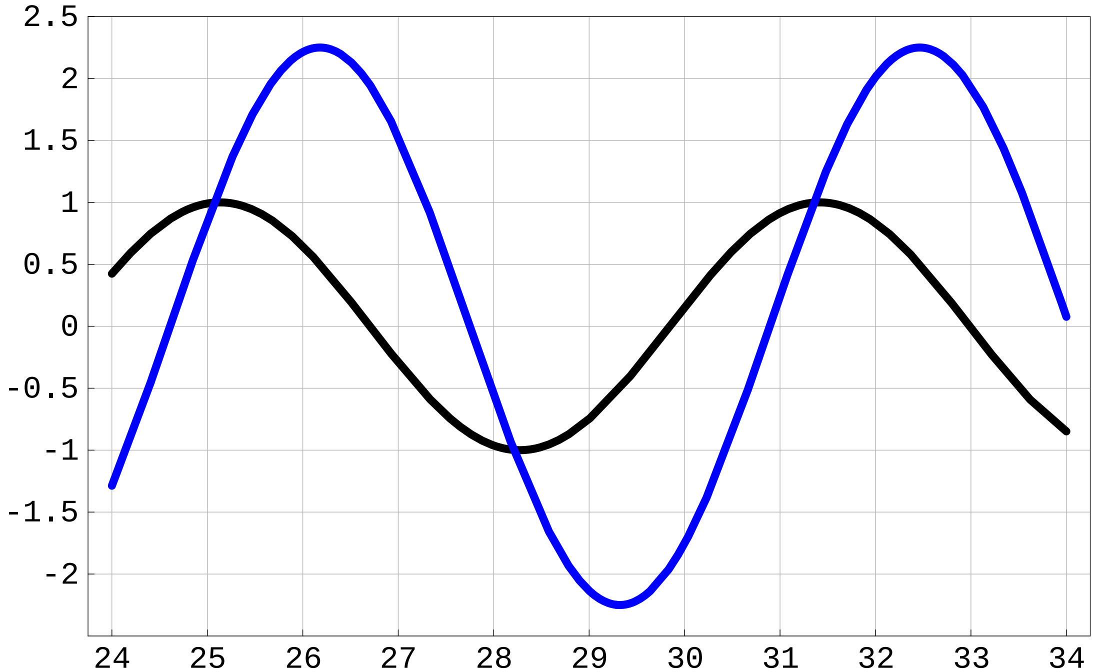

Figure 14: Problem 30

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31 and 35. The equation to be discussed is

$$x''(t) + \frac{1}{100}x'(t) + 49x(t) = A\cos(\omega t).$$

The characteristic polynomial of the equation is

$$P(\lambda) = \lambda^2 + \frac{1}{100}\lambda + 49.$$

To calculate the gain as a function of λ we calculate

$$|P(i\omega)| = \sqrt{(49-\omega^2)^2 + \left(\frac{\omega}{100}\right)^2}.$$

The gain is the reciprocal of this quantity. The maximum gain occurs at the following frequency (see formula (7.20) on page 227 in the book)

$$\omega_{\mathrm{res}} = \sqrt{49 - 2\left(\frac{1}{200}\right)^2} \approx 6.999996429.$$

The maximum gain is

$$G(\omega_{\mathrm{res}}) = \frac{1}{\sqrt{\left(49-(\omega_{\mathrm{res}})^2\right)^2 + \left(\frac{\omega_{\mathrm{res}}}{100}\right)^2}} \approx 14.2857$$

This is confirmed by the plot of the gain:

---

32 and 36. The equation to be discussed is

$$x''(t) + \frac{1}{2}x'(t) + 4x(t) = A\sin(\omega t).$$

The characteristic polynomial of the equation is

$$P(\lambda) = \lambda^2 + \frac{1}{2}\lambda + 4.$$

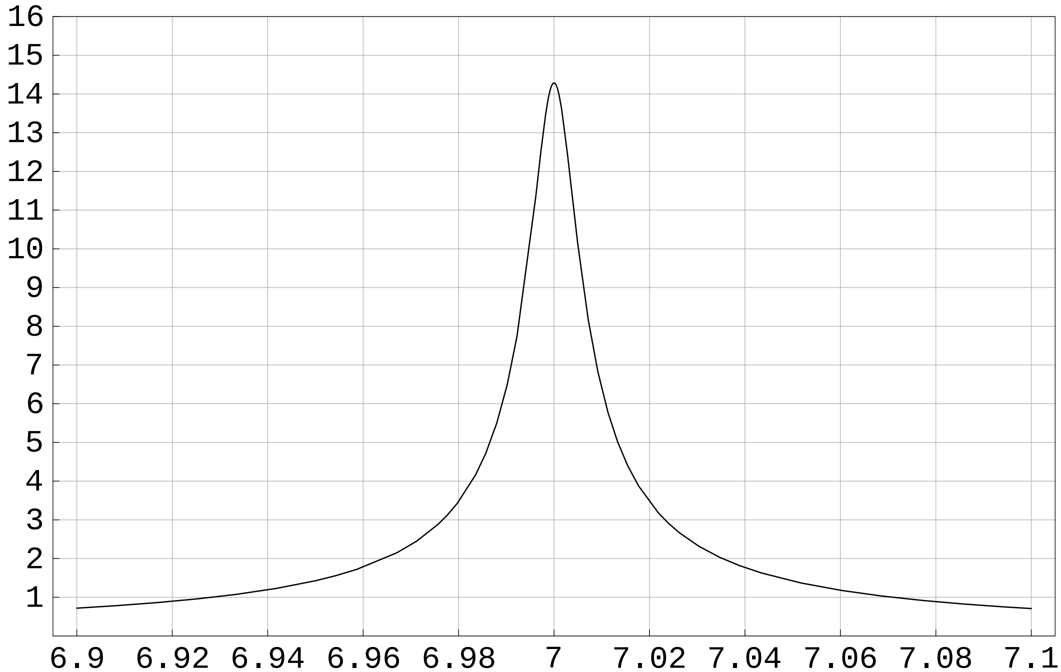

Figure 15: Problem 31, the gain as a function of ω

To calculate the gain as a function of λ we calculate

$$|P(i\omega)| = \sqrt{(4-\omega^2)^2 + \left(\frac{\omega}{2}\right)^2}.$$

The gain is the reciprocal of this quantity. The maximum gain occurs at the following frequency (see formula (7.20) on page 227 in the book)

$$\omega_{\text{res}} = \sqrt{4 - 2\left(\frac{1}{4}\right)^2} \approx 1.968501969.$$

The maximum gain is

$$G(\omega_{\text{res}}) = \frac{1}{\sqrt{\left(4-(\omega_{\text{res}})^2\right)^2 + \left(\frac{\omega_{\text{res}}}{2}\right)^2}} = \frac{8}{3\sqrt{7}} \approx 1.00791$$

The calculations are confirmed by the plot of the gain in Figure 16.

Remark. We see that the maximum gain is negligible. The reason for this is that the resistance is relatively high. In fact if the resistance was even higher, for example $2c = 1$, then the gain would be less then 1.

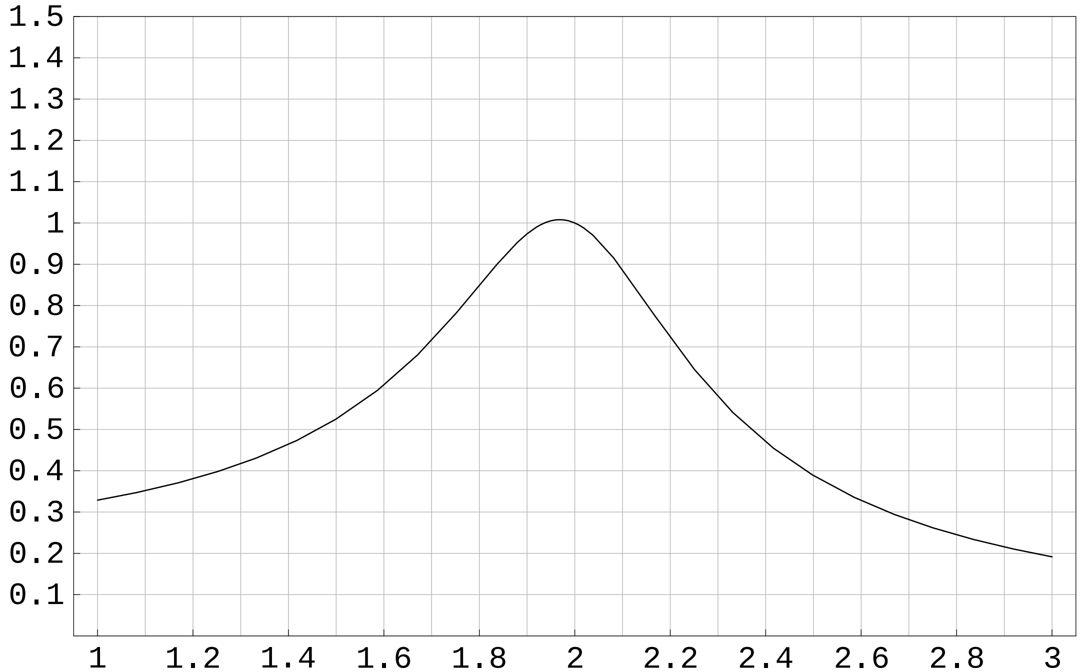

Figure 16: Problem 32, the gain as a function of ω