MATH 321

Examination 2
November 21, 2011

Name Key

Give all details of your reasoning. Each problem is 25 points for the total of 100 points.

1. (a) Transform the given initial value problem into an algebraic equation involving the Laplace transform $\mathcal{L}(y) = Y(s)$. Find the formula for $Y(s)$:

$$y''(t) + 2y'(t) + 2y(t) = e^{-t}, \qquad y(0) = 0, \quad y'(0) = 0.$$

   (b) Use partial fractions, completing the squares and properties of the Laplace transform to find the inverse Laplace transform of $\dfrac{s-1}{s^2+3s+2}$.

2. An object of mass 18kg is attached to a spring with unknown spring constant k. There is no damping present. Let $x(t)$ denote the position of the mass at time t. The mass was initially displaced by 1m from its equilibrium position and released without any initial velocity. It is observed that it took 3 seconds for the mass to reach the equilibrium for the first time. Calculate the spring constant k. (Give the exact value for k.)

3. A spring mass system is modeled by the initial value problem

$$y''(t) + 2y'(t) + 2y(t) = 0, \qquad y(0) = -2, \quad y'(0) = 0.$$

   (a) Is this spring mass system overdamped, critically damped or underdamped? Explain your answer.

   (b) Solve the given initial value problem. Denote the solution by $y(t)$.

   (c) Write the solution in the form $A\,e^{-ct}\cos(\omega_0 t - \phi)$. Here A, c, ω_0, and ϕ should be specific numbers.

   (d) Use (3c) to find the smallest positive number t_0 for which $y(t_0) = 0$. Also, give a formula for all numbers t for which $y(t) = 0$.

4. Consider the initial value problem

$$y''(t) + 16\,y(t) = 2\cos(5t), \qquad y(0) = 0, \quad y'(0) = 0.$$

   (a) Solve the given initial value problem.

   (b) Write the answer in the form $A\,\sin\!\left(\dfrac{\omega_0 - \omega}{2}\,t\right)\,\sin\!\left(\dfrac{\omega_0 + \omega}{2}\,t\right)$. Calculate the amplitude and the period of the envelope of the solution.

$\boxed{1}$

① ⓐ $s^2Y(s)+2sY(s)+2Y(s)=\frac{1}{s+1}$

$$\boxed{Y(s)=\frac{1}{(s+1)(s^2+2s+2)}}$$

ⓑ $s^2+3s+2=(s+2)(s+1)$

$$\frac{s-1}{s^2+3s+2}=\frac{s-1}{(s+2)(s+1)}=\frac{A}{s+2}+\frac{B}{s+1}$$

$$As+A+Bs+2B=s-1$$

$$A+B=1,\quad A+2B=-1$$

$$A=1-B \qquad 1-B+2B=-1$$

$$\boxed{A=3} \qquad \boxed{B=-2}$$

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s^2+3s+2}\right\}=\mathcal{L}^{-1}\left\{\frac{3}{s+2}\right\}-\mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\}$$

$$=3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}-2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}=\boxed{3e^{-2t}-2e^{-t}}$$

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② $18x''+kx(t)=0 \qquad x(0)=1,\ x'(0)=0$

$$18\lambda^2+k=0 \qquad \lambda_{1,2}=\pm\frac{\sqrt{k}}{3\sqrt{2}}=\pm\omega_0$$

$$x(t)=\underbrace{c_1}_{=1}\cos(\omega_0 t)+\underbrace{c_2}_{=0}\sin(\omega_0 t)$$

$$x(t)=\cos(\omega_0 t)$$

$$\omega_0 * 3=\frac{\pi}{2}$$

$$\frac{\sqrt{k}}{3\sqrt{2}} * 3=\frac{\pi}{2}$$

$$\sqrt{k}=\frac{\pi}{\sqrt{2}}$$

$$\boxed{k=\pi^2/2}$$

(3) $\lambda^2+2\lambda+2=0$ $\boxed{2}$

$\lambda_{1,2} = \dfrac{-2\pm\sqrt{4-8}}{2} = -1\pm i$

(a) This is underdamped system since it oscillates.

$y(t) = \underbrace{C_1}_{-2} e^{-t}\cos t + C_2 e^{-t}\sin t$

$y(t) = -2e^{-t}\cos t + C_2 e^{-t}\sin t$

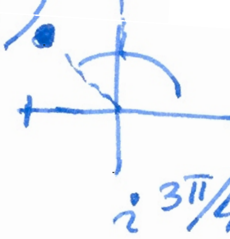
$y'(t) = -e^{-t}(-2\cos t + C_2\sin t) + e^{-t}(2\sin t + C_2\cos t)$

$0 = y'(0) = 2 + C_2, \quad C_2 = -2$

(b) $\boxed{y(t) = -2e^{-t}(\cos t + \sin t)}$

$y'(t) = 2e^{-t}(c(t)+s(t)) - 2e^{-t}(-s(t)+c(t))$

(c) $y(t) = 2e^{-t}\cos(t\ldots)$.



$-2e^{-t}(e^{it} + ie^{it}) =$

$= 2e^{-t}e^{it}(-1+i) = 2e^{-t}e^{it}\sqrt{2}\,e^{i3\pi/4}$

$= 2\sqrt{2}\,e^{-t}e^{i(t+3\pi/4)}$

(c) $y(t) = 2\sqrt{2}\, e^{-t} \cos(t + 3\pi/4)$ $\boxed{3}$

$A = 2\sqrt{2}$ $\cancel{e^{-t}}$ $\boxed{c = -1}$

$\phi = -3\pi/4$

(d) $\cos(\underbrace{t + 3\pi/4}) = 0$

$= \pi/2 \quad t = -\frac{\pi}{4}$

$= 3\pi/2 \quad t = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

$\boxed{t_0 = \frac{3\pi}{4}}$ $\quad t = \frac{3\pi}{4} + k\pi$

where $k = 0, \pm 1, \ldots$

④ ⓐ

$$\lambda^2+16=0$$

$$\lambda=\pm 4$$

$$y_H(t)=c_1\cos(4t)+c_2\sin(4t)$$

$$y_p(t)=\dots$$

$$z_p(t)=a e^{i5t}$$

$$z''(t)+16z=2e^{i5t}$$

$$-25a\cancel{e^{i5t}}+16a\,a\cancel{e^{i5t}}=2e^{i5t}$$

$$\boxed{a=-\frac{2}{9}}$$

$$z_p(t)=-\frac{2}{9}e^{5it}$$

$$\boxed{y_p(t)=-\frac{2}{9}\cos(5t)}$$

$$y(t)=\underset{2/9}{c_1}\cos(4t)+\underset{0}{\underset{\shortparallel}{c_2}}\sin(4t)-\frac{2}{9}\cos(5t)$$

ⓑ $y(t)=\frac{2}{9}\left(\cos(4t)-\cos(5t)\right)$

$\boxed{4}$

(b) $y(t) = \frac{4}{9} \sin\left(\frac{t}{2}\right) \sin\left(\frac{9t}{2}\right)$ $\boxed{5}$

amplitude of the envelope is $4/9$ and the period of the envelope is $T: \frac{T}{2} = 2\pi$

$T = 4\pi.$