Section 2.3 version October 5, 2011 at 18:16

Assigned problems: 1-6, 8, 10, 12. The answers are rounded to five significant digits.

1. $t = 6.1224 \times 10^6$

2. $t = 6.3004$

Solution. The initial value problem for the velocity is

$$m\frac{dv}{dt} = -mg, \quad v(0) = 15.$$

The solution is $v(t) = -gt + 15$. Now, the initial value problem for the position is

$$\frac{dx}{dt} = -gt + 15, \quad x(0) = 100.$$

The solution is $x(t) = -\frac{g}{2}t^2 + 15t + 100$. The solution is found by solving $x(t) = 0$ for $t > 0$:

$$\frac{5\left(3 + \sqrt{8g + 9}\right)}{g} \approx 6.30041.$$

3. It took the stone 7.24378 seconds to reach the bottom of the well which is 257.115 meters deep.

Solution. The initial value problem for the velocity is

$$m\frac{dv}{dt} = -mg, \quad v(0) = 0.$$

The solution is $v(t) = -gt$. Denote by d the depth of the well and put the origin at the bottom of the well. Now, the initial value problem for the position is

$$\frac{dx}{dt} = -gt, \quad x(0) = d.$$

The solution is $x(t) = -\frac{g}{2}t^2 + d$. Denote by τ the time it took the stone to hit the bottom of the well, that is

$$d = \frac{g}{2}\tau^2.$$

It is given that

$$\tau + \frac{d}{340} = 8.$$

Therefore

$$340(8 - \tau) = \frac{g}{2}\tau^2.$$

Solving the last equation for $0 < \tau < 8$ we get

$$\tau = \frac{4\left(-85 + \sqrt{85}\sqrt{4g + 85}\right)}{g} \approx 7.24378.$$

Now, $d = 340(8 - \tau) \approx 257.115$.

4. maximum altitude 2.0167×10^6m $= 2,016.7$km, total time 1313.79 seconds or 21 minutes and 53.79 seconds.

Solution. There are two parts of this problem. During the first 6o seconds the initial value problem for the velocity is

$$\frac{dv}{dt} = 100, \quad v(0) = 0.$$

The solution is $v(t) = 100t$. The initial value problem for the position is

$$\frac{dx}{dt} = 100t, \quad x(0) = 0.$$

The solution is $x(t) = 50t^2$. Record the values at $t = 60$:

$$v(60) = 6 \times 10^3, \quad x(60) = 18 \times 10^4.$$

After 60 seconds the initial value problem for the velocity is

$$\frac{dv}{dt} = -g, \quad v(60) = 6 \times 10^3.$$

The solution is $v(t) = g(60 - t) + 6000$. At this point we can calculate the time t_0 at which the altitude will reach the maximum.

$$t_0 = 60 + \frac{6 \times 10^3}{g} \approx 672.245.$$

The initial value problem for the position is

$$\frac{dx}{dt} = g(60 - t) + 6 \times 10^3, \quad x(60) = 18 \times 10^4.$$

The solution is

$$x(t) = -\frac{g}{2}(60 - t)^2 + 6 \times 10^3\, t - 18 \times 10^4.$$

Now calculating $x(t_0)$ yields the maximum altitude

$$\begin{aligned}
x(t_0) &= -\frac{g}{2}\frac{36 \times 10^6}{g^2} + 6 \times 10^3 \left(60 + \frac{6 \times 10^3}{g}\right) - 18 \times 10^4 \\
&= -\frac{18 \times 10^6}{g} + 18 \times 10^4 + \frac{36 \times 10^6}{g} \\
&= 18 \times 10^4 \left(\frac{100}{g} + 1\right) \\
&\approx 2.01673 \times 10^6
\end{aligned}$$

Solving $x(t) = 0$ for $t > t_0$ yields the time when the rocket returns to the earth:

$$t = \frac{60}{g}\left(100 + g + 10\sqrt{100 + g}\right) \approx 1313.7887.$$

In Figure 1 the altitude of the rocket while the motor is working is marked by red, while the altitude of the rocket after the motor is shut off is marked by red. In Figure 1 the units on the time axes are in minutes and the units of altitude are in kilometers.

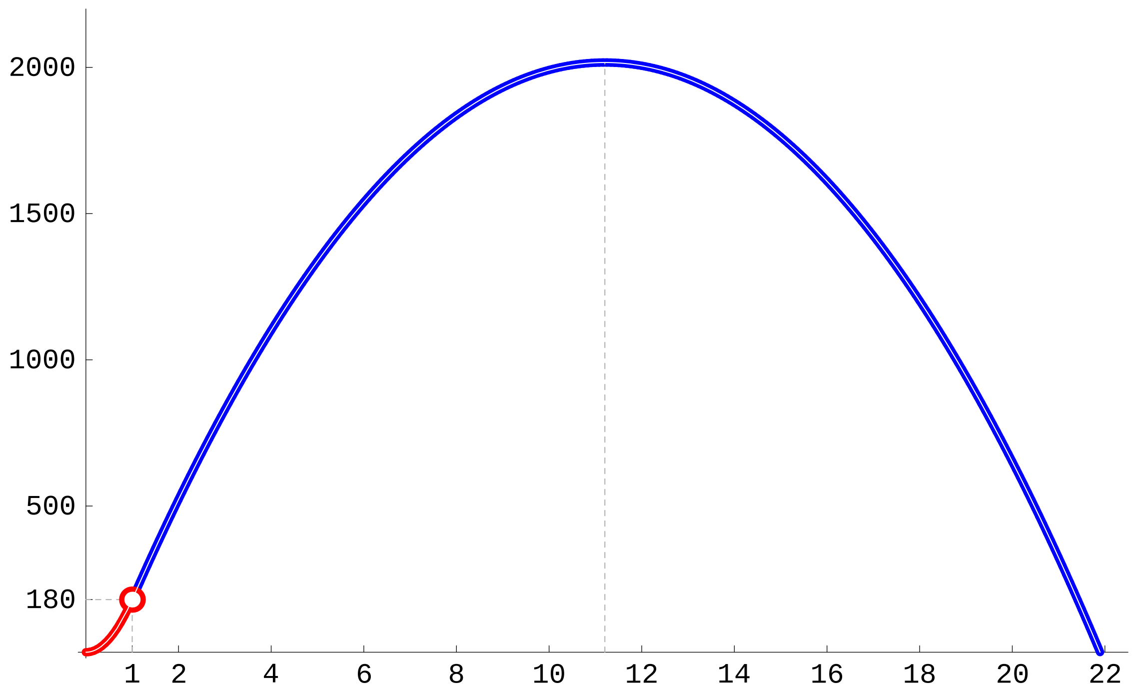

Figure 1: Problem 4

5. It took 3.4142 seconds to complete the distance of 57.119 meters.
Solution. Denote by x_0 the initial hight of the body and denote by τ the time it took for the body to fall to the ground. Then the initial value problem for the velocity is

$$m\frac{dv}{dt} = -mg, \quad v(0) = 0,$$

with the solution $v(t) = -gt$. The initial value problem for the position is

$$\frac{dx}{dt} = -gt, \quad x(0) = x_0,$$

with the solution

$$x(t) = x_0 - \frac{g}{2}t^2.$$

It is given that

$$x_0 = \frac{g}{2}\tau^2 \quad \text{and} \quad \frac{x_0}{2} = x_0 - \frac{g}{2}(\tau-1)^2 = x_0 - \frac{g}{2}\tau^2 + g\tau - \frac{g}{2} = g\tau - \frac{g}{2}.$$

Substituting the firs equality in the second we get

$$\frac{g}{4}\tau^2 = g\tau - \frac{g}{2}.$$

Simplifying yields

$$\tau^2 - 4\tau + 2 = 0.$$

Solving this equation for $\tau > 1$ yields $\tau = 2 + \sqrt{2} \approx 3.41421$. Now it is easy to calculate x_0:

$$x_0 = \frac{g}{2}\tau^2 = g\big(3 + 2\sqrt{2}\big) \approx 57.1186.$$

6. (a) $\dfrac{{v_0}^2}{2\,g}$ (b) It takes $\dfrac{v_0}{g}$ seconds to reach the maximum height. It takes $\dfrac{v_0}{g}$ seconds, the same time, for the ball to return to the ground. The reason is that the height of the ball is expressed as a quadratic function. The graph of this function is a parabola. The t-intercepts of this parabola are symmetric with respect to its vertex. (c) The speed of the ball on its return to the ground is v_0.

8. To reach one half of the terminal velocity it will take $\dfrac{m \ln 2}{r}$, where m is the mass of the ball and r is the constant of proportionality in the expression for the resistance. During this time the ball will travel $\dfrac{g\,m^2\,(\ln(4)-1)}{2\,r^2}$.

10. (a) The velocity at the end of of 2 seconds is $-20\left(1-e^{-g/10}\right) \approx -12.494$ m/s, while the distance traveled is $-40\left(\dfrac{10}{g}\left(1-e^{-g/10}\right)-1\right) \approx 14.502$ m. (b) $\dfrac{20}{g}\ln(5) \approx 3.2846$ seconds.

12. -17.34 m/s

Solution. The initial value problem for the velocity is

$$mv' = -mg - rv, \quad v(0) = v_0.$$

In this particular problem $m = 1/5$ kg, $r = 1/10$ and $v_0 = 0$. It might be easier to solve the general equation and then use the specific given values. Rewrite the equation as

$$v' = -\frac{r}{m}\left(\frac{mg}{r} + v\right), \quad v(0) = v_0,$$

and separate the variables

$$\frac{1}{\frac{mg}{r} + v} v' = -\frac{r}{m}.$$

Now integrate

$$\int \frac{1}{\frac{mg}{r} + v} dv = \ln\left(\frac{mg}{r} + v\right).$$

This gives

$$\frac{d}{dt}\left(\ln\left(\frac{mg}{r} + v(t)\right)\right) = \frac{1}{\frac{mg}{r} + v} v'(t).$$

Therefore the equation

$$\frac{1}{\frac{mg}{r} + v} v' = -\frac{r}{m}$$

can be rewritten as

$$\frac{d}{dt}\left(\ln\left(\frac{mg}{r} + v(t)\right)\right) = -\frac{r}{m},$$

and the last equation has the solution

$$\ln\left(\frac{mg}{r} + v(t)\right) = -\frac{r}{m}t + C.$$

Now use the initial condition to solve for C:

$$\ln\left(\frac{mg}{r} + v(t)\right) = -\frac{r}{m}t + \ln\left(\frac{mg}{r} + v_0\right).$$

Finally, solve for $v(t)$:

$$\frac{mg}{r} + v(t) = \left(\frac{mg}{r} + v_0\right) e^{-\frac{r}{m}t},$$

that is

$$v(t) = \left(\frac{mg}{r} + v_0\right) e^{-\frac{r}{m}t} - \frac{mg}{r}.$$

The initial value problem for the position $x(t)$ is

$$x'(t) = \left(\frac{mg}{r} + v_0\right) e^{-\frac{r}{m}t} - \frac{mg}{r}, \quad x(0) = x_0.$$

The solution of the differential equation is

$$x(t) = -\frac{m}{r}\left(\frac{mg}{r} + v_0\right) e^{-\frac{r}{m}t} - \frac{mg}{r}t + C.$$

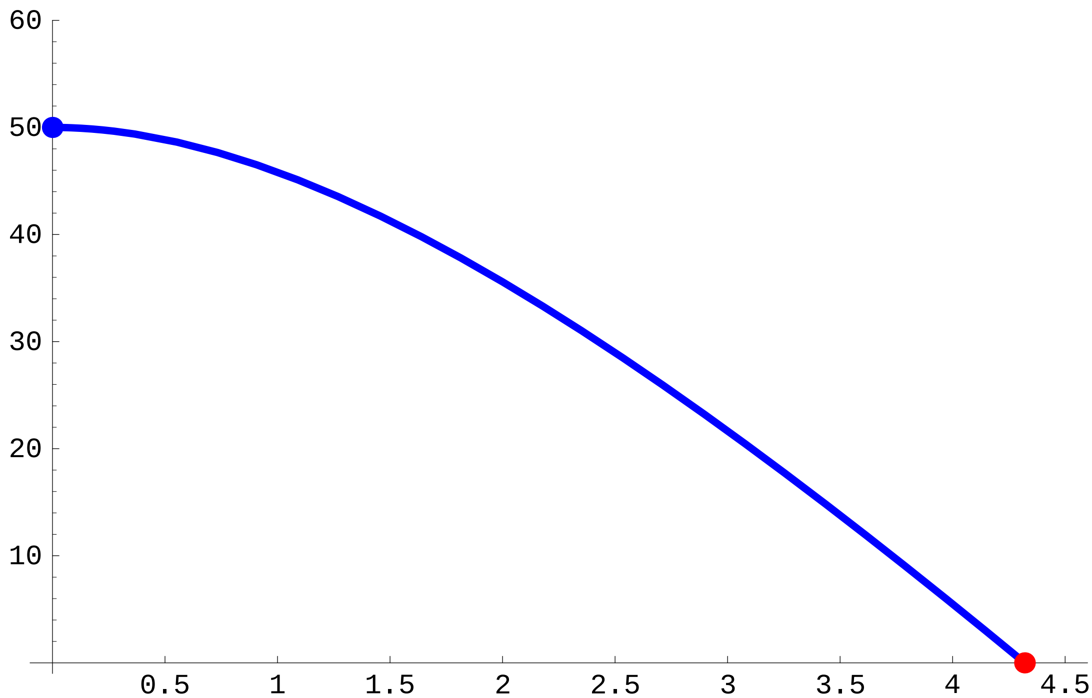

Figure 2: Problem 12

Hence the solution of the initial value problem is

$$x(t) = -\frac{m}{r}\left(\frac{mg}{r} + v_0\right)\left(e^{-\frac{r}{m}t} - 1\right) - \frac{mg}{r}t + x_0.$$

In our particular case $m/r = 2$, $v_0 = 0$ and $x_0 = 50$. Therefore,

$$x(t) = -4g\left(e^{-t/2} - 1\right) - 2gt + 50.$$

With $g = 9.8$ the plot of this function is given in Figure 2
To find the velocity of the mass when it hits the ground, we must solve for t the equation

$$-4 * 9.8\left(e^{-t/2} - 1\right) - 2 * 9.8\, t + 50 = 0.$$

An approximate solution of this equation found by Mathematica command

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FindRoot[-4*9.8 (Exp[-t/2] - 1) - 2*9.8 t + 50 == 0, {t,4}]
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is $t_0 \approx 4.32042$. To solve the exercise we substitute this value in our solution for the velocity:

$$v(t) = 2 * 9.8\left(e^{-t/2} - 1\right)$$

to get $v(t_0) \approx -17.3401$.

14.

Solution. By Newton's law of gravitation the force exerted on the object in Figure 1 in the book is

$$-\frac{GMm}{(R+y)^2}.$$

Here G is the universal gravitational constant, M is the mass of the earth, R is the approximate radius of the earth, m is the mass of the object in Figure 1 in the book and y is the distance of the object from the earth's surface. The air resistance is ignored. But, we know that for $y=0$, that is on the earth's surface this force is $-mg$, where $g \approx 9.8$m/s is earth's gravitational acceleration. Therefore,

$$-\frac{GMm}{R^2} = -mg, \qquad \text{that is} GM = gR^2.$$

The standard approximation for R is 6371km.

If an object with mass m is launched from the earth's surface with initial velocity v_0 the resulting initial value problem for its velocity is

$$m\frac{d}{dt}v(t) = -\frac{gR^2m}{(R+y(t))^2}, \quad v(0) = v_0.$$

The difficulty in the last equation is that we have two dependent variables v and y. In fact $v(t) = y'(t)$, thus $v't = y''(t)$ so the above equation is a second order equation for y. But, we can consider v as a composite function, $v(t) = v(y(t))$. (Here we assume that at each position y there is only one velocity, which really is not true. What is true that at each position there can be two velocities, one positive and one negative. By considering only nonnegative velocities we eliminate a possible confusion.) Then by the chain rule

$$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}.$$

Substituting the lase expression in the initial value problem for velocity we get

$$v\frac{dv}{dy} = -\frac{gR^2}{(R+y)^2}, \quad v(0) = v_0.$$

This is a separable equation in which v is the unknown function of y. Since

$$\int v dv = \frac{1}{2}v^2$$

we have

$$\frac{d}{dy}\frac{1}{2}(v(y))^2 = v(y)\frac{dv}{dy}.$$

Therefore the differential equation in the initial value problem above becomes

$$\frac{d}{dy}\frac{1}{2}(v(y))^2 = -\frac{gR^2}{(R+y)^2}$$

Solving the last equation yields

$$\frac{1}{2}(v(y))^2 = -\int \frac{gR^2}{(R+y)^2}\,dy = \frac{gR^2}{R+y} + C.$$

Using the initial condition $v(0) = v_0$ and solving for C we get

$$v^2 = v_0^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{R+y}\right)$$

Since we consider only nonnegative velocities we have

$$v(y) = \sqrt{v_0^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{R+y}\right)}.$$

Now the question is: What is the interval of existence of this solution? For the solution to be defined we must have

$$v_0^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{R+y}\right) \geq 0.$$

The last inequality can be rewritten as

$$v_0^2 - 2gR \geq -\frac{2gR^2}{R+y}.$$

Since $y \geq 0$ the quantity $-2gR^2/(R+y)$ is always negative. Therefore, if $v_0^2 - 2gR \geq 0$ the interval of existence of the solution is $y \geq 0$. This means that if $v_0^2 \geq 2gR$ the velocity of the object will always remain positive and the object will keep moving away from the Earth. The smallest velocity for this happens is

$$v_0 = \sqrt{2gR} \approx 11174.6\text{m/s} \approx 11.2\text{km/s}.$$

This velocity is called *escape velocity.*
If $v_0^2 - 2gR < 0$, then the interval of existence is obtained by solving for $y > 0$ the following equation

$$v_0^2 - 2gR = -\frac{2gR^2}{R+y}.$$

The solution is

$$y = \frac{Rv_0^2}{2gR - v_0^2} > 0.$$

Thus the interval of existence is

$$0 \leq y \leq \frac{Rv_0^2}{2gR - v_0^2}.$$

The quantity $\frac{Rv_0^2}{2gR - v_0^2}$ is the maximum height reached by the object. At the maximum height the velocity of the object is 0.

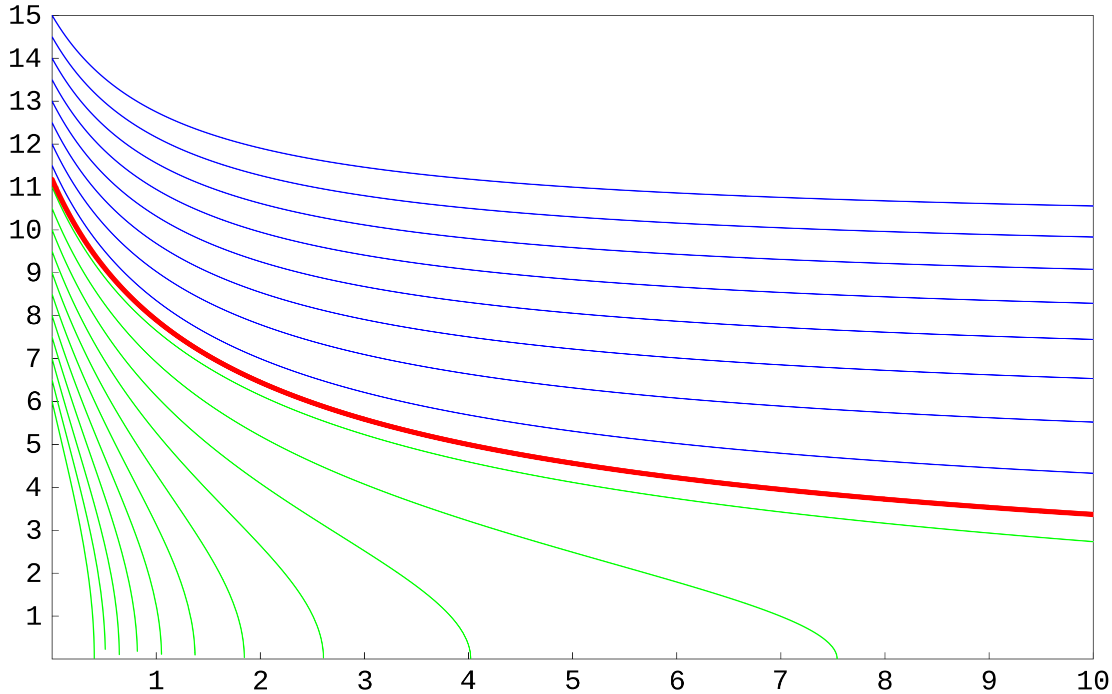

Figure 3: Problem 14

In Figure 3 I plotted many solutions of the initial value problem considered here. In Figure 3 the horizontal axis is y, the distance from the earth's surface. The unit along the horizontal axis is R, earth's radius. The vertical axis is v, the velocity of the object; the units are km/s. I plotted the solutions for $v_0 = 6, 6.5, 7, 7.5, 8, 8.5, 9, 9.5, 10, 10.5, 11$km/s in green. Each of these solutions has a finite interval of existence; that is the object reaches a maximum height. For $v_0 = 11$km/s the maximum height is approximately $31.25 \times R$.
The red solution is the solution for which $v_0 \approx \sqrt{2gR} \approx 11.2$km/s. Since this is the escape velocity this solution is defined for all $y \geq 0$, the object does never return to the Earth. The blue solutions have the initial velocities $v_0 = 11.5, 12, 12.5, 13, 13.5, 14, 14.5, 15$km/s. Each of these velocities is larger than escape velocity, so the solutions are also defined for all $y \geq 0$.