Section 4.5 version November 16, 2011 at 23:01

Exercises

2. The given equation is

$$y''(t) + 6y'(t) + 8y(t) = -3e^{-t}.$$

To find a particular solution we try $y(t) = ae^{-t}$ and calculate

$$ae^{-t} - 6ae^{-t} + 8ae^{-t} = -3e^{-t}.$$

Thus $3a = -3$, that is $a = -1$. Thus the particular solution is

$$y_p(t) = -e^{-t}.$$

Finding the general solution of the corresponding homogeneous equation we have the general solution of this equation is

$$y(t) = C_1 e^{-2t} + C_2 e^{-4t} - e^{-t}.$$

5. The given equation is

$$y''(t) + 4y(t) = \cos(3t).$$

Instead we solve the complex equation

$$z''(t) + 4z(t) = e^{3i\,t}.$$

Then the real part of the solution of the complex equation will be the solution of the given equation.

For the complex equation try $a\,e^{3i\,t}$. Find the first and the second derivative and substitute in the equation:

$$-9a\,e^{3i\,t} + 4a\,e^{3i\,t} = e^{3i\,t}.$$

Thus we need to solve $-5a = 1$. Thus a particular solution of the complex equation is

$$z(t) = -\frac{1}{5}\,e^{3i\,t} = \frac{1}{5}\,e^{(3t+\pi)\,i} = \frac{1}{5}\big(\cos(3t+\pi) + i\sin(3t+\pi)\big).$$

The real part of the complex solution is a solution of the original equation

$$y_p(t) = \frac{1}{5}\,\cos(3t+\pi).$$

I prefer this way of writing solution since it emphasizes the amplitude as a positive number 1/5 and it shows the phase $\phi = -\pi$. This solution is identical to the solution in the book.

The general solution of the given differential equation is

$$y(t) = C_1\cos(2t) + C_2\sin(2t) + \frac{1}{5}\,\cos(3t+\pi).$$

6. The given equation is

$$y''(t) + 9y(t) = \sin(2t).$$

Instead we solve the complex equation

$$z''(t) + 9z(t) = e^{2i\,t}.$$

Then the imaginary part of the solution of the complex equation will be the solution of the given equation.
For the complex equation try $a\,e^{2i\,t}$. Find the first and the second derivative and substitute in the equation:

$$-4a\,e^{2i\,t} + 9a\,e^{2i\,t} = e^{2i\,t}.$$

Thus we need to solve $5a = 1$. Thus a particular solution of the complex equation is

$$z(t) = \frac{1}{5}\,e^{2i\,t} = \frac{1}{5}\big(\cos(2t) + i\sin(2t)\big).$$

The imaginary part of the complex solution is a solution of the original equation

$$y_p(t) = \frac{1}{5}\,\sin(2t).$$

The general solution of the given differential equation is

$$y(t) = C_1\cos(3t) + C_2\sin(3t) + \frac{1}{5}\,\sin(2t).$$

BC comment. The above equation can be viewed as a model of a spring-mass system without damping and with the forcing term (in my picture drawn on the blackboard it is "wind") $\sin(2t)$. Now we can answer the question how this system will respond if the mass is at rest at time $t = 0$.
From $y(0) = 0$ we conclude that $C_1 = 0$. Since we also have $y'(0) = 0$ we have $3C_2 + 2/5 = 0$. Hence $c_2 = -2/15$. Thus the solution is (see Figure 1)

$$y(t) = \frac{1}{15}\big(-2\,\sin(3t) + 3\,\sin(2t)\big).$$

The period of $\sin(3t)$ is $2\pi/3$ and the period of $\sin(2t)$ is π. Thus, the period of the solution is 2π.
If you are wondering how other solutions look like look at Figures 2 and 3. In Figure 2 I vary the initial position and in Figure 3 I vary the initial velocity.

8. The given equation is

$$y''(t) + 7y'(t) + 10y(t) = -4\sin(3t).$$

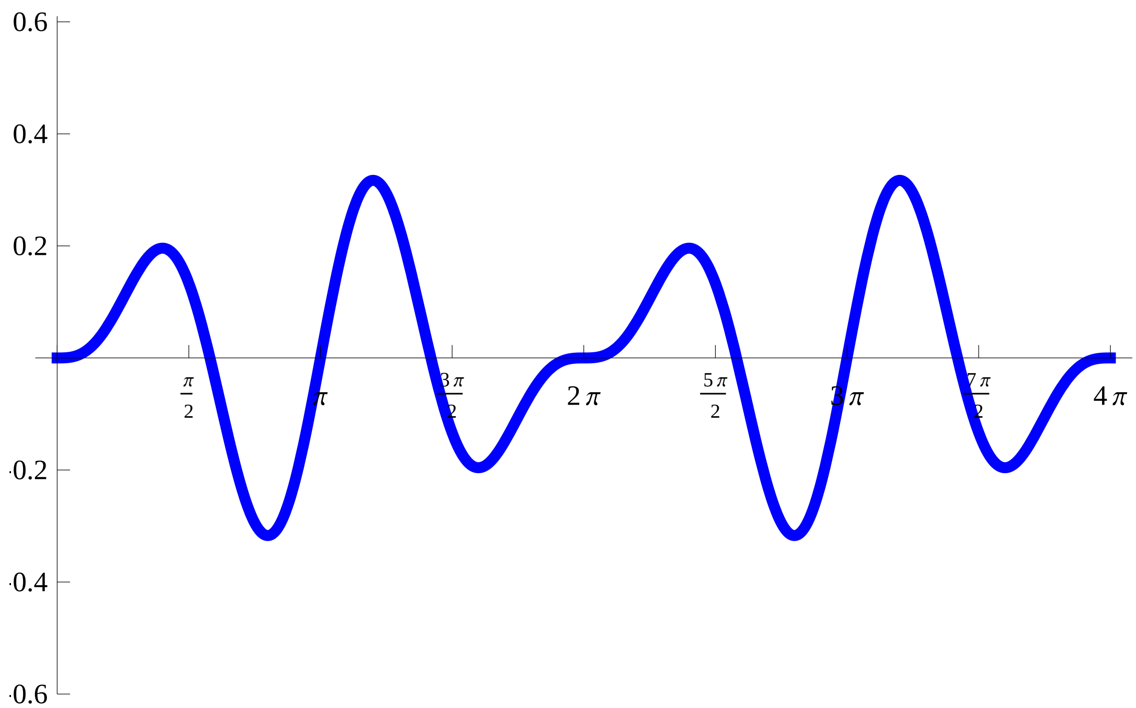

Figure 1: Problem 6

Instead we solve the complex equation

$$z''(t) + 7z'(t) + 10z(t) = -4\, e^{3t\, i}.$$

Then the imaginary part of the solution of the complex equation will be the solution of the given equation.

For the complex equation try $a\, e^{3i\, t}$. Find the first and the second derivative and substitute in the equation:

$$-9a\, e^{3i\, t} + 21\, a\, i\, e^{3i\, t} + 10\, a\, e^{3i\, t} = -4\, e^{3i\, t}.$$

Thus we need to solve $a(1 + 21\, i) = -4$. Thus a particular solution of the complex equation is

$$z_p(t) = \frac{-4}{1 + 21\, i}\; e^{3i\, t} = \frac{-4}{1 + 21\, i}\; \bigl(\cos(3t) + i \sin(3t)\bigr) = \frac{-4 + 84\, i}{442}\; \bigl(\cos(3t) + i \sin(3t)\bigr).$$

The imaginary part of the last displayed function is

$$y_p(t) = \frac{84}{442} \cos(3t) - \frac{4}{442} \sin(3t) = \frac{2}{221}\bigl(21 \cos(3t) - \sin(3t)\bigr)$$

This is a particular solution.

But, it is always good to find a solution in the amplitude-phase form. For that write $1 + 21\, i$ in the polar form:

$$1 + 21\, i = \sqrt{442}\, e^{\phi\, i}, \qquad \text{where} \qquad \phi = \arctan(21)\,.$$

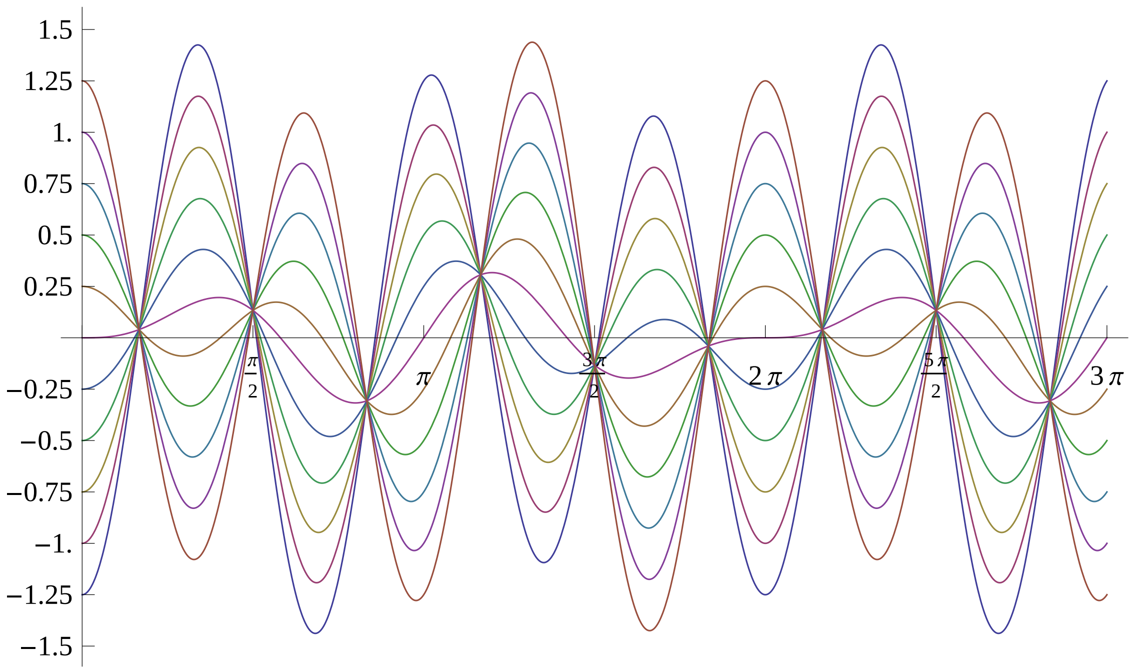

Figure 2: Problem 6

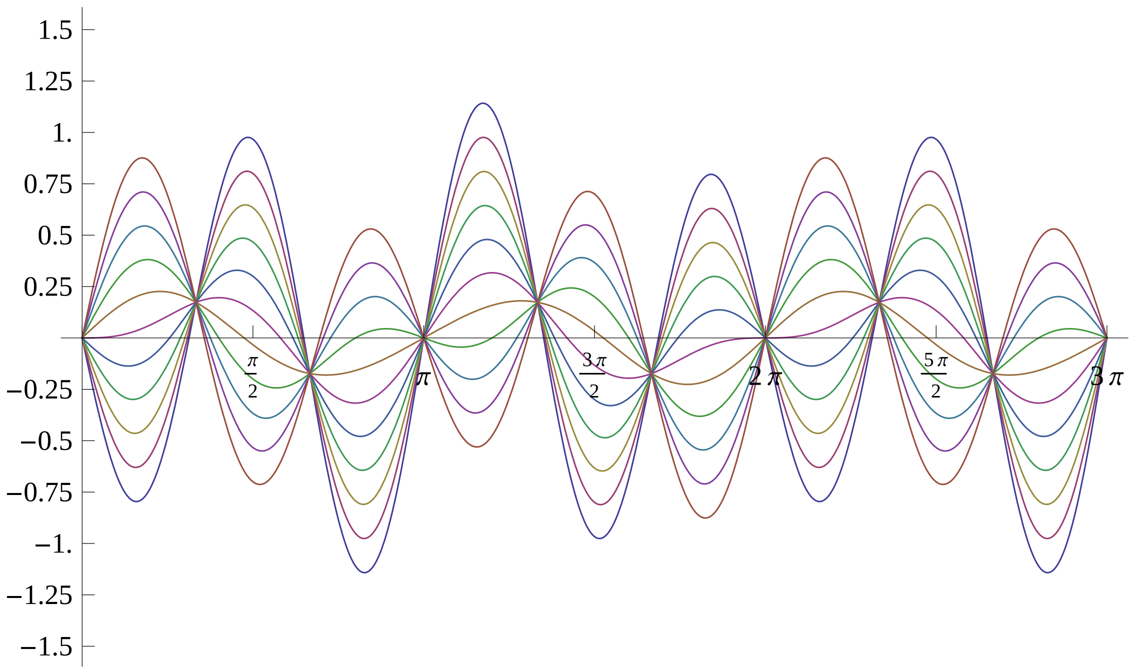

Figure 3: Problem 6

Then,

$$\frac{1}{1+21\,i} = \frac{1}{\sqrt{442}}\,e^{-\phi\,i}, \qquad \text{where} \qquad \phi = \arctan(21)\,,$$

and consequently, since $-1 = e^{\pi i}$, we have

$$z_p(t) = \frac{-4}{\sqrt{442}}\, e^{-\phi i}\, e^{3it} = \frac{4}{\sqrt{442}}\, e^{(\pi-\phi)i}\, e^{3it} = \frac{4}{\sqrt{442}}\, e^{(3t+\pi-\phi)i}.$$

The imaginary part here is

$$y_p(t) = \frac{4}{\sqrt{442}}\, \sin(3t + \pi - \phi) = \frac{4}{\sqrt{442}}\, \sin\bigl(3t - (\phi - \pi)\bigr).$$

Just to verify that two formulas that we found coincide we show them both in Figure 4.

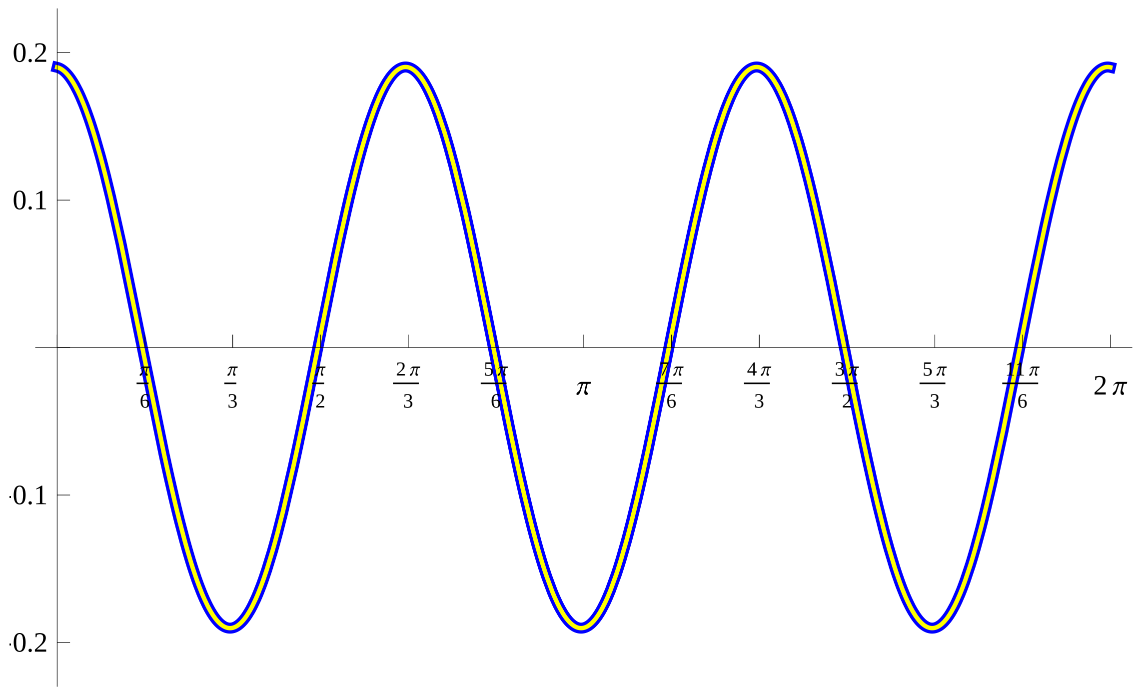

Figure 4: Problem 8

21. The initial value problem to be solved is

$$y''(t) - 2y'(t) + 5y(t) = 3\cos(t), \qquad y(0) = 0, \quad y'(0) = -2.$$

The characteristic equation of the corresponding homogeneous equation is

$$\lambda^2 - 2\lambda + 5 = 0.$$

The solution of this equation are

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2\,i$$

Thus the general solution of the corresponding homogeneous equation is

$$y_H(t) = C_1 e^t \cos(2t) + C_2 e^t \sin(2t).$$

To find a particular solution we solve the complex equation:

$$z''(t) - 2z'(t) + 5z(t) = 3e^{it}.$$

We look for a solution in the form $a\,e^{it}$. Substituting yields

$$-a\,e^{it} - 2\,i\,a\,e^{it} + 5\,a\,e^{it} = 3e^{it}.$$

To find a we solve $a(4 - 2\,i) = 3$. Thus the solution is

$$z_p(t) = \frac{3}{4 - 2\,i} e^{it} = \frac{12 + 6\,i}{20}\big(\cos(t) + i\,\sin(t)\big)$$

The real part of the last function is

$$y_p(t) = \frac{3}{5}\cos(t) - \frac{3}{10}\sin(t)$$

Thus the general solution of the given equation is

$$y(t) = C_1 e^t \cos(2t) + C_2 e^t \sin(2t) + \frac{3}{5}\cos(t) - \frac{3}{10}\sin(t)$$

To solve for C_1 and C_2 we use initial conditions: $y(0) = 0$ and $y'(0) = -2$. The first condition $y(0) = 0$ yields $C_1 = -3/5$. Now we calculate $y'(t)$:

$$y'(t) = e^t\big(C_1\,\cos(2t) + C_2\,\sin(2t)\big) + e^t\big(-2C_1\,\sin(2t) + 2C_2\,\cos(2t)\big) - \frac{3}{5}\sin(t) - \frac{3}{10}\cos(t).$$

Then

$$y'(0) = C_1 + 2C_2 - \frac{3}{10} = -2.$$

Hence $C_2 = -11/20$ and the solution of the given problem is

$$y(t) = -\frac{3}{5} e^t \cos(2t) - \frac{11}{20} e^t \sin(2t) + \frac{3}{5}\cos(t) - \frac{3}{10}\sin(t).$$

26. The given equation is

$$y''(t) + 4y(t) = 4\cos(2t).$$

Instead we solve the complex equation

$$z''(t) + 4z(t) = 4e^{2it}.$$

Since $\mathrm{Re}\big(4e^{2i\,t}\big) = 4\cos(2t)$, the real part of the solution of the complex equation will be the solution of the given equation.
For the complex equation try $a\,e^{2i\,t}$. Find the first and the second derivative and substitute in the equation:

$$-4a\,e^{2i\,t} + 4a\,e^{2i\,t} = 4e^{2i\,t}.$$

Thus we get $0 = 4e^{2i\,t}$, which is impossible. So, we have to change the guessed solution to $z(t) = a\,t\,e^{2i\,t}$. Then calculate

$$\begin{aligned} z'(t) &= a\,e^{2i\,t} + 2ia\,t\,e^{2i\,t} = a\big(1 + 2i\,t\big)e^{2i\,t}, \\ z''(t) &= 2i\,ae^{2i\,t} + 2i\,a\big(1 + 2i\,t\big)e^{2i\,t} = 4a\big(i - t\big)\,e^{2i\,t} \end{aligned}$$

Substitute the second derivative in the equation and we get

$$4a\big(i - t\big)\,e^{2i\,t} + 4\,a\,t\,e^{2i\,t} = 4e^{2i\,t}.$$

Simplifying yields

$$4aie^{2i\,t} = 4e^{2i\,t},$$

that is $ai = 1$, or $a = -i$. Thus, the complex solution is

$$z(t) = -i\,t\,e^{2i\,t} = -it\,\cos(2t) + t\,\sin(2t).$$

Since we are looking for the real part, a particular solution of the given equation is

$$t\,\sin(2t).$$

The general solution is

$$y(t) = C_1\cos(2t) + C_2\sin(2t) + t\,\sin(2t).$$

This solution is significantly different than any solutions. The function $t\,\sin(2t)$ (see Figure 5) is unbounded. In terms of a string: if the forces are aligned as in this problem, the string would break. This phenomenon is called resonance.
Now we can find the solution that satisfies $y(0) = 0$ and $y'(0) = 0$. That solution is $t\,\sin(2t)$ and it is shown in Figure 5. Several other solutions are shown in Figures 6 and 7.

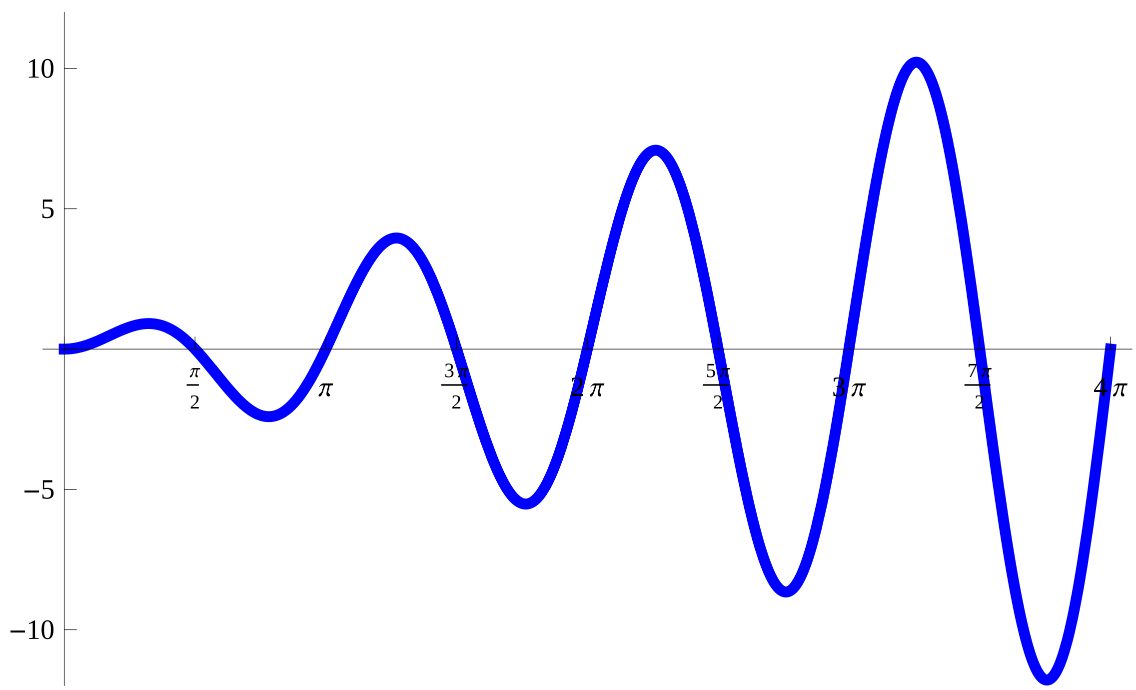

Figure 5: Problem 26

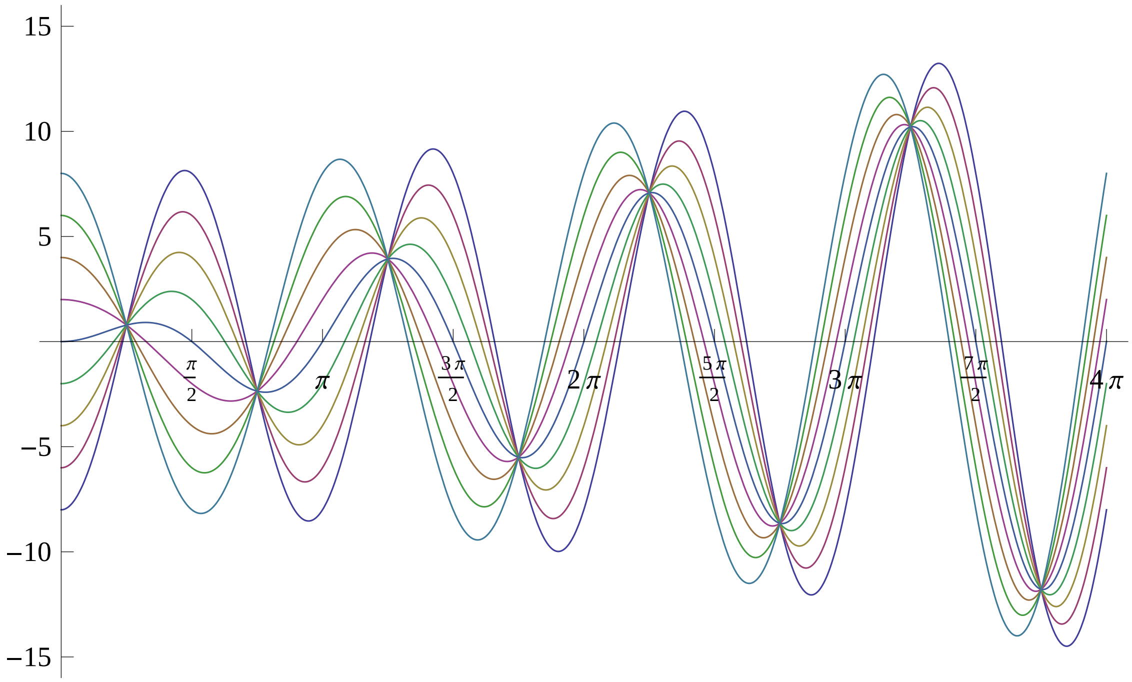

Figure 6: Problem 26

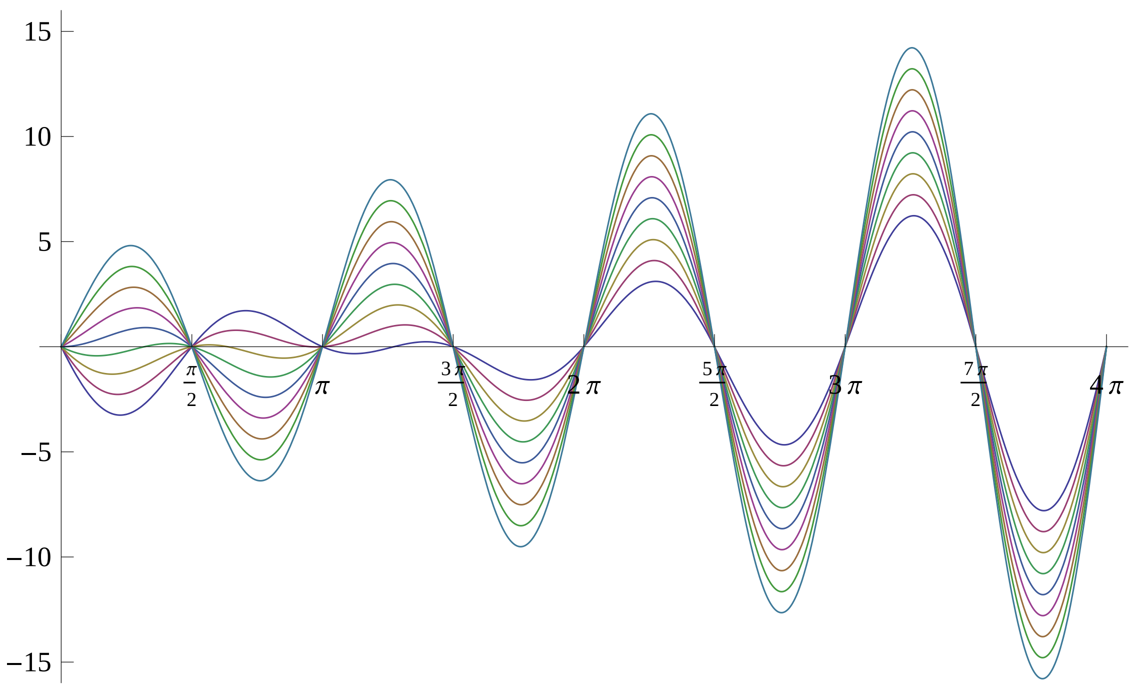

Figure 7: Problem 26