Section 5.5 version December 4, 2011 at 13:57

Exercises

6. We need to calculate the Laplace transform of

$$H(t-2)\, e^{-t}.$$

Notice that the Laplace transform of

$$H(t-2)\, e^{-(t-2)} = H(t-2)\, f(t-2) \qquad \text{where} \qquad f(t) = e^{-t}.$$

can be calculated using the Table of Laplace transforms. Thus the Laplace transform of

$$H(t-2)\, e^{-(t-2)} \qquad \text{is} \qquad e^{-2s}\, \frac{1}{s+1}.$$

But,

$$H(t-2)\, e^{-(t-2)} = H(t-2)\, e^{-t+2} = e^{2}\, H(t-2)\, e^{-t},$$

and therefore

$$H(t-2)\, e^{-t} = e^{-2}\, H(t-2)\, e^{-(t-2)}.$$

Thus the Laplace transform of

$$H(t-2)\, e^{-t} \qquad \text{is} \qquad e^{-2}\, e^{-2s}\, \frac{1}{s+1} = e^{-2(s+1)}\, \frac{1}{s+1}.$$

13. The given function is (see Figure 1)

$$f(t) = \begin{cases} 0, & \text{if} \quad t < 0; \\ t^2, & \text{if} \quad 0 \le t < 2; \\ 4, & \text{if} \quad 2 \le t \end{cases}$$

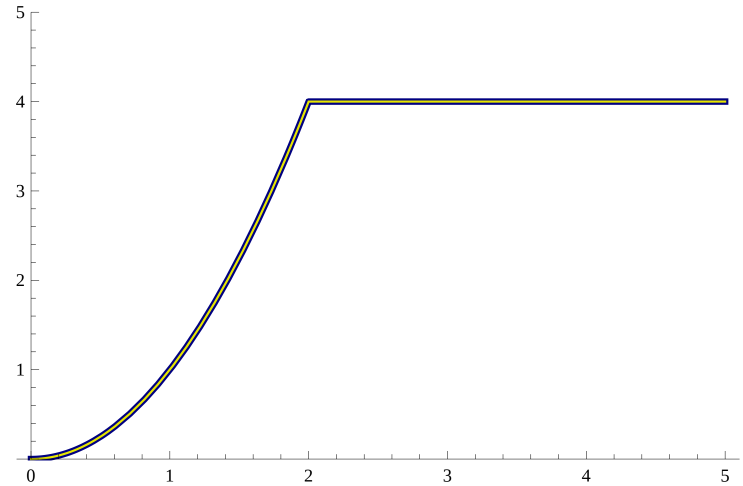

Figure 1: Problem 13

Using the interval function $H_{ab}(t)$ we can write $f(t)$ as

$$\begin{aligned} f(t) &= t^2\, H_{0,2}(t) + 4\, H_2(t) \\ &= t^2\, \big(H_0(t) - H_2(t)\big) + 4\, H_2(t) \\ &= t^2\, H_0(t) - t^2 H_2(t) + 4\, H_2(t) \\ &= t^2 + \big(4 - t^2\big) H_2(t) \\ &= t^2 + H(t-2)\big(-4(t-2) - (t-2)^2\big) \\ &= t^2 - 4(t-2)H(t-2) - (t-2)^2 H(t-2) \end{aligned}$$

Now we can use the table of the transforms to find the Laplace transform of $f(t)$

$$\frac{2}{s^3} - 4\,\frac{e^{-2s}}{s^2} - 2\,\frac{e^{-2s}}{s^3} = \frac{2}{s^3}\big(1 - e^{-2s}\big) - e^{-2s}\,\frac{4}{s^2}.$$

14. The given function is (see Figure 2)

$$f(t) = \begin{cases} 3, & \text{if} \quad 0 \le t < 1; \\ 2, & \text{if} \quad 1 \le t < 2; \\ 1, & \text{if} \quad 2 \le t < 3; \\ 0, & \text{if} \quad 3 \le t \end{cases}$$

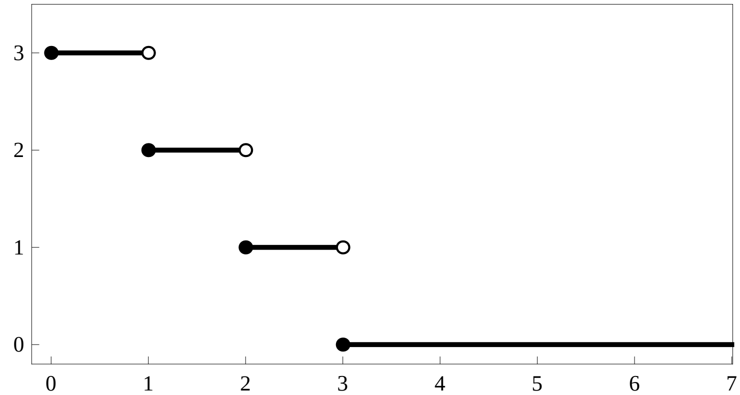

Figure 2: Problem 14

Using the interval function $H_{ab}(t)$ we can write $f(t)$ as

$$\begin{aligned} f(t) &= 3\,H_{0,1}(t) + 2\,H_{1,2}(t) + 1\,H_{2,3}(t) \\ &= 3\big(H_0(t) - H_1(t)\big) + 2\,\big(H_1(t) - H_2(t)\big) + \big(H_2(t) - H_3(t)\big) \\ &= 3\,H_0(t) - H_1(t) - H_2(t) - H_3(t) \\ &= 3 - H_1(t) - H_2(t) - H_3(t). \end{aligned}$$

Now we can use the table of the transforms to find the Laplace transform of $f(t)$

$$\frac{3}{s} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} = \frac{3 - e^{-s} - e^{-2s} - e^{-3s}}{s}.$$

22. The given function is

$$F(s) = \frac{1 - e^{-s}}{s(s+2)} = \frac{1}{s(s+2)} - e^{-s}\frac{1}{s(s+2)}.$$

To find the inverse laplace transform of this function first find the inverse Laplace transform of

$$F_1(s) = \frac{1}{s(s+2)} = \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right).$$

From the table of Laplace transforms we see that it is the function

$$f_1(t) = \frac{1}{2}\big(1 - e^{-2t}\big).$$

Now looking at the table again we can see that the inverse laplace transform of this function first find the inverse Laplace transform of

$$F_2(s) = e^{-s}\frac{1}{s(s+2)}$$

is

$$f_2(t) = H(t-1)f(t-1) = \frac{1}{2}\,H(t-1)\left(1 - e^{-2(t-1)}\right) = \frac{1}{2}\,H(t-1)\left(1 - e^{2}\,e^{-2t}\right)$$

Finally, the inverse Laplace transform of the given function is

$$f(t) = f_1(t) - f_2(t) = \frac{1}{2}\big(1 - e^{-2t}\big) - \frac{1}{2}\,H(t-1)\left(1 - e^{2}\,e^{-2t}\right).$$

It is interesting to rewrite this function $f(t)$ as a piecewise defined function:

$$f(t) = \begin{cases} \frac{1}{2}\big(1 - e^{-2t}\big) & \text{if} \quad 0 \le t < 1; \\ \frac{1}{2}\left(e^{2} - 1\right)e^{-2t} & \text{if} \quad 1 \le t \end{cases}$$

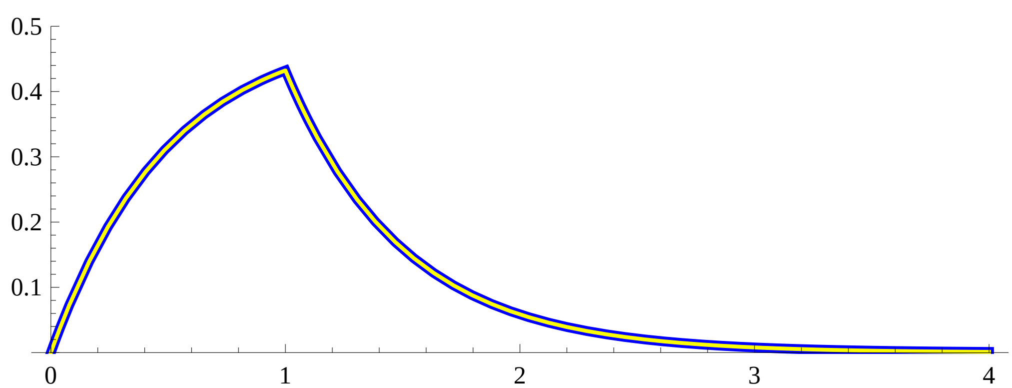

Figure 3: Problem 22

28. The initial value problem that we are asked to solve can be written as

$$y''(t) + 4y(t) = H_1(t) - H_2(t), \qquad y(0) = 0, \;\; y'(0) = 0.$$

Applying the Laplace transform to both sides of the equation and using the initial conditions we get the equation for $Y(s)$:

$$(s^2 + 4)Y(s) = \frac{e^{-s} - e^{-2s}}{s}.$$

The solution is

$$Y(s) = \frac{e^{-s} - e^{-2s}}{s(s^2 + 4)} = e^{-s}\,\frac{1}{s(s^2+4)} - e^{-2s}\,\frac{1}{s(s^2+4)}.$$

To calculate the inverse Laplace transform of $Y(s)$ we first calculate the inverse Laplace transform of

$$\frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right).$$

That is the function

$$\frac{1}{4}\,(1 - \cos(2t))$$

Using the table of Laplace transform we read that the inverse Laplace transform of

$$e^{-s}\,\frac{1}{s(s^2+4)} \qquad \text{is} \qquad \frac{1}{4}\,H(t-1)\,\big((1 - \cos(2(t-1))\big)$$

and the inverse Laplace transform of

$$e^{-2s}\,\frac{1}{s(s^2+4)} \qquad \text{is} \qquad \frac{1}{4}\,H(t-2)\,\big(1 - \cos(2(t-2))\big).$$

Therefore

$$y(t) = \frac{1}{4}\Big(H(t-1)\,\big(1 - \cos(2t-2)\big) - H(t-2)\,\big(1 - \cos(2t-4)\big)\Big).$$

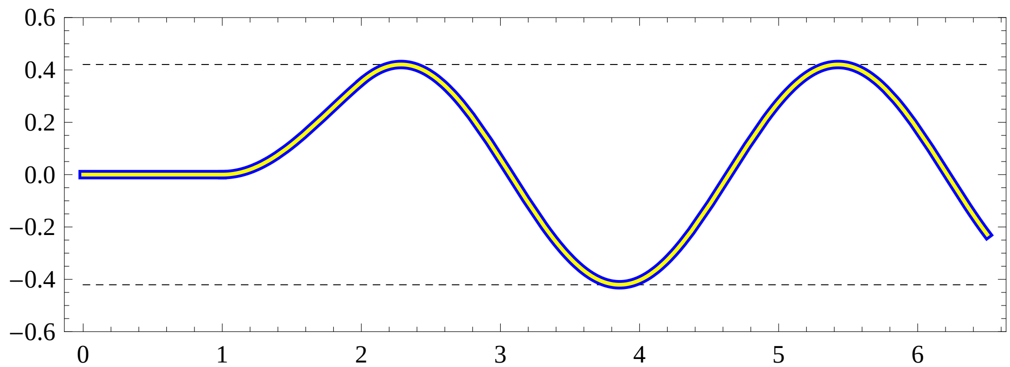

Figure 4: Problem 28

Again, we rewrite $y(t)$ as a piecewise defined function:

$$y(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < 1; \\ \frac{1}{4}\big(1 - \cos(2t-2)\big) & \text{if} \quad 1 \leq t < 2 \\ \frac{1}{4}\big(\cos(2t-4) - \cos(2t-2)\big) & \text{if} \quad 2 \leq t \end{cases}$$

The expression $\cos(2t-4) - \cos(2t-2)$ can be simplified using Euler's identity

$$\begin{aligned} \cos(2t-4) - \cos(2t-2) &= \operatorname{Re}\big(e^{(2t-4)i} - e^{(2t-2)i}\big) \\ &= \operatorname{Re}\big(e^{(2t-3)\,i}\big(e^{-i} - e^{i}\big)\big) \\ &= \operatorname{Re}\big(e^{(2t-3)\,i}\big(\cos(1) - i\sin(1) + \cos(1) - i\sin(1)\big)\big) \\ &= \operatorname{Re}\big(e^{(2t-3)\,i}\big(-2i\sin(1)\big)\big) \\ &= 2\sin(1)\sin(2t-3). \end{aligned}$$

Thus the amplitude of the third part of the solution is $\dfrac{\sin(1)}{2}$. This is indicated by dashed lines in Figure 4.

A periodic on-off switch

Let c be a positive real number. A periodic on-off switch is a periodic function with period $2c$ with the graph given in Figure 5 below.

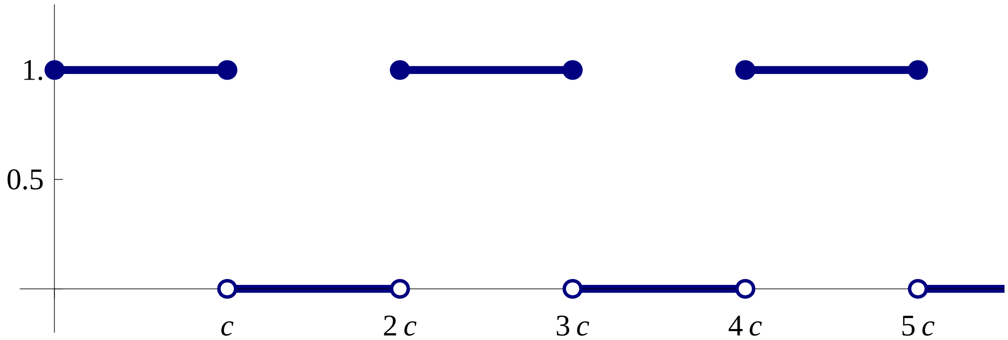

Figure 5: A periodic on-off switch

A possible formula for this function is $f(t) = H\big(\sin(\pi t/c)\big)$. For example, for $c = 1.5$ the graph is given in Figure 6.

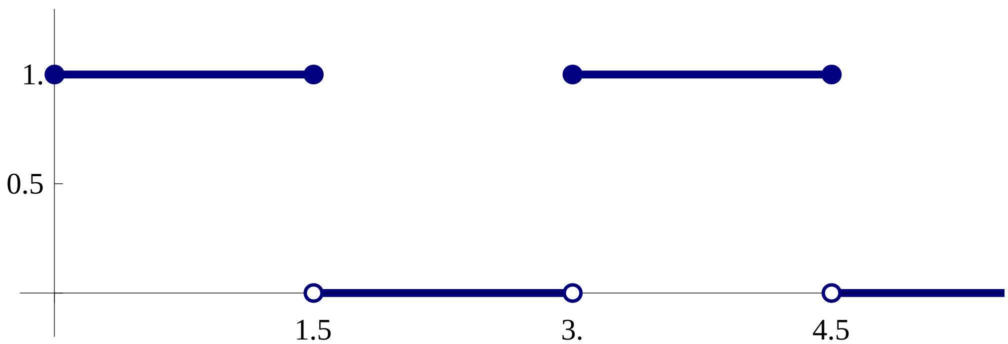

Figure 6: A periodic on-off switch, $c = 1.5$

The window of the function $f(t)$ (with unspecified c) is the function $1 - H_c(t)$. The graph is given in Figure 7 where the graph of the window is in blue with the dark blue part belonging to $f(t)$ and the light blue part is the part outside of the interval $[0, 2c)$.

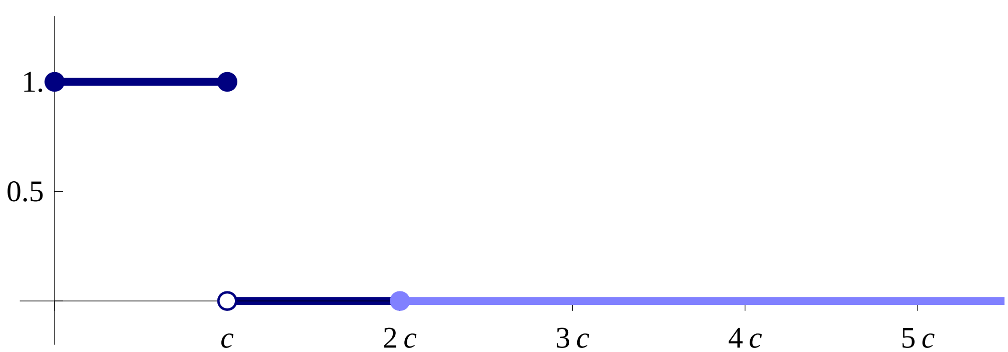

Figure 7: The window of a periodic on-off switch

The Laplace transform of the window of $f(t)$, that is of $1 - H_c(t)$, is $\dfrac{1-e^{-cs}}{s}$. Therefore the Laplace transform of the periodic on-off switch is

$$\frac{1-e^{-cs}}{s\big(1-e^{-2cs}\big)} = \frac{1}{s\big(1+e^{-cs}\big)}$$