MATH 430

Assignment 2
November 20, 2012

Name ________________________________

Problem 1. Consider the function $x \mapsto x^2$, $0 < x < \pi$. Denote by f the Fourier periodic extension of the even extension of this function. Denote by g the Fourier periodic extension of the odd extension of the given function. Denote by h the Fourier periodic extension of the function obtained by extending the given function by 0 on $(-\pi, 0]$.

(a) Find the Fourier series of the functions f, g and h.

(b) What is the relationship between the functions f, g and h? Explain how this relationship could be used to simplify the calculations in (a).

(c) Find the following sums $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. Explain your answer using the Fourier series calculated above.

(d) Use the properties of the translation and the addition formula for cosine to find which function has $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx)$ as its Fourier series. Use Mathematica to confirm that your solution is correct.

(e) Performing term by term indefinite integration of the Fourier series in (d), find which function has $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} \cos(nx)$ as its Fourier series. Mathematica can be helpful here, but I expect a derivation of the formula that is not Mathematica dependent.

(f) In (e) you found the sums of some interesting numerical series. Which ones?

Problem 2. Consider the function $x \mapsto e^{ax}$, $-\pi < x < \pi$.

(a) Find the Fourier series of this function. Express all the Fourier coefficients using the hyperbolic (not the exponential!) functions.

(b) Use the Fourier series from (a) to find the sums $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a^2 + n^2}$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{a^2 + n^2}$.

(c) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}$ converge uniformly on $\mathbb{R}$? Explain the best you can. Even a fully rigorous explanation is not too difficult.

Problem 3. Consider the vibrating string equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$$

subject to the boundary conditions

$$u(0,t) - \frac{\partial u}{\partial x}(0,t) = 0, \qquad 2\,u(\pi,t) + \frac{\partial u}{\partial x}(\pi,t) = 0, \quad t \geq 0,$$

and the initial conditions

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 \leq x \leq \pi.$$

(a) Solve the above problem.

(b) This problem can be interpreted as a mathematical model of vibrations of a string whose endpoints are fixed and the end parts of the string are rigid (soaked in super-glue). Assume that the in initial shape of the string is $f(x) = \frac{1}{5}\left(-1 - x - \left(\pi + \frac{1}{2} - \frac{3}{2\pi}\right)x^2 + x^3\right), 0 \leq x \leq \pi$, and assume that $f(x)$ satisfies the boundary conditions. What is the domain of the entire string? Write the equation for the initial shape of the entire string?

(c) Illustrate in Mathematica.

Problem 4. This problem is for graduate students only. Consider a string of length 5, that is $0 \leq x \leq 5$, which is fixed at its endpoints. Assume the the part of the string between 1 and 2 is rigid (soaked in super-glue).

(a) Establish a mathematical model for the vibrations of this string.

(b) Solve the problem that you formulated in (a).

(c) Illustrate in Mathematica with the initial shape given by $f(x) = 3x - 2x^2, 0 \leq x \leq 1$, $f(x) = 2 - x, 1 < x < 2$ and $f(x) = (10 - 7x + x^2)/3, 2 \leq x \leq 5$, and no initial velocity.

(d) Illustrate in Mathematica with no initial displacement and the initial velocity $g(x) = \sin(2\pi x/5), 0 \leq x \leq 5$.