

Chapter 2

Method of Separation of Variables

2.1 Introduction

In Chapter 1 we developed from physical principles an understanding of the heat equation and its corresponding initial and boundary conditions. We are ready to pursue the mathematical solution of some typical problems involving partial differential equations. We will use a technique called the method of separation of variables. You will have to become an expert in this method, and so we will discuss quite a few examples. We will emphasize problem solving techniques, but we must also understand how not to misuse the technique.

A relatively simple but typical, problem for the equation of heat conduction occurs for a one-dimensional rod ($0 \leq x \leq L$) when all the thermal coefficients are constant. Then the PDE,

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \frac{Q(x, t)}{c\rho}, \quad \begin{array}{l} t > 0 \\ 0 < x < L, \end{array} \quad (2.1.1)$$

must be solved subject to the initial condition,

$$u(x, 0) = f(x), \quad 0 < x < L, \quad (2.1.2)$$

and two boundary conditions. For example, if both ends of the rod have prescribed temperature, then

$$\begin{array}{l} u(0, t) = T_1(t) \\ u(L, t) = T_2(t). \end{array} \quad \begin{array}{l} t > 0 \\ \end{array} \quad (2.1.3)$$

The method of separation of variables is used when the partial differential equation and the boundary conditions are linear and homogeneous, concepts we now explain.

2.2 Linearity

As in the study of ordinary differential equations, the concept of linearity will be very important for us. A **linear operator** L by definition satisfies

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2) \quad (2.2.1)$$

for any two functions u_1 and u_2 , where c_1 and c_2 are arbitrary constants. $\partial/\partial t$ and $\partial^2/\partial x^2$ are examples of linear operators since they satisfy (2.2.1):

$$\begin{aligned} \frac{\partial}{\partial t}(c_1 u_1 + c_2 u_2) &= c_1 \frac{\partial u_1}{\partial t} + c_2 \frac{\partial u_2}{\partial t} \\ \frac{\partial^2}{\partial x^2}(c_1 u_1 + c_2 u_2) &= c_1 \frac{\partial^2 u_1}{\partial x^2} + c_2 \frac{\partial^2 u_2}{\partial x^2}. \end{aligned}$$

It can be shown (see Exercise 2.2.1) that any linear combination of linear operators is a linear operator. Thus, the **heat operator**

$$\frac{\partial}{\partial t} - k \frac{\partial^2}{\partial x^2}$$

is also a linear operator.

A **linear equation** for u it is of the form

$$L(u) = f, \quad (2.2.2)$$

where L is a linear operator and f is known. Examples of *linear partial differential equations* are

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + f(x, t) \quad (2.2.3)$$

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \alpha(x, t)u + f(x, t) \quad (2.2.4)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (2.2.5)$$

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + \alpha(x, t)u. \quad (2.2.6)$$

Examples of *nonlinear* partial differential equations are

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \alpha(x, t)u^4 \quad (2.2.7)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{\partial^3 u}{\partial x^3}. \quad (2.2.8)$$

The u^4 and $u\partial u/\partial x$ terms are nonlinear; they do not satisfy (2.2.1).

If $f = 0$, then (2.2.2) becomes $L(u) = 0$, called a **linear homogeneous** equation. Examples of linear homogeneous partial differential equations include the heat equation,

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = 0, \quad (2.2.9)$$

as well as (2.2.5) and (2.2.6). From (2.2.1) it follows that $L(0) = 0$ (let $c_1 = c_2 = 0$). Therefore, $u = 0$ is always a solution of a linear homogeneous equation. For example, $u = 0$ satisfies the heat equation (2.2.9). We call $u = 0$ the *trivial* solution of a linear homogeneous equation. The simplest way to test whether an equation is homogeneous is to substitute the function u identically equal to zero. If $u \equiv 0$ satisfies a linear equation, then it must be that $f = 0$ and hence the linear equation is homogeneous. Otherwise, the equation is said to be **nonhomogeneous** [e.g., (2.2.3) and (2.2.4)].

The fundamental property of linear operators (2.2.1) allows solutions of linear equations to be added together in the following sense:

Principle of Superposition

If u_1 and u_2 satisfy a linear *homogeneous* equation, then an arbitrary linear combination of them, $c_1 u_1 + c_2 u_2$, also satisfies the same linear homogeneous equation.

The proof of this depends on the definition of a linear operator. Suppose that u_1 and u_2 are two solutions of a linear homogeneous equation. That means that $L(u_1) = 0$ and $L(u_2) = 0$. Let us calculate $L(c_1 u_1 + c_2 u_2)$. From the definition of a linear operator,

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2).$$

Since u_1 and u_2 are homogeneous solutions, it follows that $L(c_1 u_1 + c_2 u_2) = 0$. This means that $c_1 u_1 + c_2 u_2$ satisfies the linear homogeneous equation $L(u) = 0$ if u_1 and u_2 satisfy the *same* linear homogeneous equation.

The concepts of linearity and homogeneity also apply to boundary conditions, in which case the variables are evaluated at specific points. Examples of *linear* boundary conditions are the conditions we have discussed:

$$u(0, t) = f(t) \quad (2.2.10)$$

$$\frac{\partial u}{\partial x}(L, t) = g(t) \quad (2.2.11)$$

$$\frac{\partial u}{\partial x}(0, t) = 0 \quad (2.2.12)$$

$$-K_0 \frac{\partial u}{\partial x}(L, t) = h[u(L, t) - g(t)]. \quad (2.2.13)$$

A *nonlinear* boundary condition, for example, would be

$$\frac{\partial u}{\partial x}(L, t) = u^2(L, t). \quad (2.2.14)$$

Only (2.2.12) is satisfied by $u \equiv 0$ (of the linear conditions) and hence is homogeneous. It is not necessary that a boundary condition be $u(0, t) = 0$ for $u \equiv 0$ to satisfy it.

EXERCISES 2.2

2.2.1. Show that any linear combination of linear operators is a linear operator.

2.2.2. (a) Show that $L(u) = \frac{\partial}{\partial x} [K_0(x) \frac{\partial u}{\partial x}]$ is a linear operator.

(b) Show that usually $L(u) = \frac{\partial}{\partial x} [K_0(x, u) \frac{\partial u}{\partial x}]$ is not a linear operator.

2.2.3. Show that $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q(u, x, t)$ is linear if $Q = \alpha(x, t)u + \beta(x, t)$ and in addition homogeneous if $\beta(x, t) = 0$.

2.2.4. In this exercise we derive superposition principles for nonhomogeneous problems.

(a) Consider $L(u) = f$. If u_p is a particular solution, $L(u_p) = f$, and if u_1 and u_2 are homogeneous solutions, $L(u_i) = 0$, show that $u = u_p + c_1 u_1 + c_2 u_2$ is another particular solution.

(b) If $L(u) = f_1 + f_2$ where u_{pi} is a particular solution corresponding to f_i , what is a particular solution for $f_1 + f_2$?

2.2.5 If L is a linear operator, show that $L\left(\sum_{n=1}^M c_n u_n\right) = \sum_{n=1}^M c_n L(u_n)$. Use this result to show that the principle of superposition may be extended to any finite number of homogeneous solutions.

2.3 Heat Equation with Zero Temperatures at Finite Ends

2.3.1 Introduction

Partial differential equation (2.1.1) is linear but it is homogeneous only if there are no sources, $Q(x, t) = 0$. The boundary conditions (2.1.3) are also linear, and they too are homogeneous only if $T_1(t) = 0$ and $T_2(t) = 0$. We thus first propose to study

$$\text{PDE: } \boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}} \quad \begin{array}{l} 0 < x < L \\ t > 0 \end{array} \quad (2.3.1)$$

$$\text{BC: } \boxed{\begin{array}{l} u(0, t) = 0 \\ u(L, t) = 0 \end{array}} \quad (2.3.2)$$

$$\text{IC: } \boxed{u(x, 0) = f(x)}. \quad (2.3.3)$$

The problem consists of a linear homogeneous partial differential equation with linear homogeneous boundary conditions. There are two reasons for our investigating this type of problem, (2.3.1)-(2.3.3), besides the fact that we claim it can be solved by the method of separation of variables. First, this problem is a relevant physical problem corresponding to a one-dimensional rod ($0 < x < L$) with no sources and both ends immersed in a 0° temperature bath. We are very interested in predicting how the initial thermal energy (represented by the initial condition) changes in this relatively simple physical situation. Second, it will turn out that in order to solve the nonhomogeneous problem (2.1.1)-(2.1.3), we will need to know how to solve the homogeneous problem, (2.3.1)-(2.3.3).

2.3.2 Separation of Variables

In the **method of separation of variables**, we attempt to determine solutions in the product form

$$u(x, t) = \phi(x)G(t), \quad (2.3.4)$$

where $\phi(x)$ is only a function of x and $G(t)$ only a function of t . Equation (2.3.4) must satisfy the *linear homogeneous* partial differential equation (2.3.1) and boundary conditions (2.3.2), but for the moment we set aside (ignore) the initial condition. The **product solution**, (2.3.4), does not satisfy the initial conditions. Later we will explain how to satisfy the initial conditions.

Let us be clear from the beginning — we do not give any reasons why we choose the form (2.3.4). (Daniel Bernoulli invented this technique in the 1700s. *It works*, as we shall see.) We substitute the assumed product form, (2.3.4), into the partial differential equation (2.3.1):

$$\begin{aligned} \frac{\partial u}{\partial t} &= \phi(x) \frac{dG}{dt} \\ \frac{\partial^2 u}{\partial x^2} &= \frac{d^2 \phi}{dx^2} G(t), \end{aligned}$$

and consequently the heat equation (2.3.1) implies that

$$\phi(x) \frac{dG}{dt} = k \frac{d^2 \phi}{dx^2} G(t). \quad (2.3.5)$$

We note that we can “separate variables” by dividing both sides of (2.3.5) by $\phi(x)G(t)$:

$$\frac{1}{G} \frac{dG}{dt} = k \frac{1}{\phi} \frac{d^2 \phi}{dx^2}.$$

Now the variables have been “separated” in the sense that the left-hand side is only a function of t and the right-hand side only a function of x . We can continue in this way, but it is *convenient* (i.e., not necessary) also to divide by the constant k ,

and thus

$$\underbrace{\frac{1}{kG} \frac{dG}{dt}}_{\text{function of } t \text{ only}} = \underbrace{\frac{1}{\phi} \frac{d^2 \phi}{dx^2}}_{\text{function of } x \text{ only}}. \quad (2.3.6)$$

This could be obtained directly from (2.3.5) by dividing by $k\phi(x)G(t)$. How is it possible for a function of time to equal a function of space? If x and t are both to be arbitrary independent variables, then x cannot be a function of t (or t a function of x) as seems to be specified by (2.3.6). The important idea is that we claim it is necessary that both sides of (2.3.6) must equal the same constant:

$$\frac{1}{kG} \frac{dG}{dt} = \frac{1}{\phi} \frac{d^2 \phi}{dx^2} = -\lambda, \quad (2.3.7)$$

where λ is an arbitrary *constant* known as the **separation constant**.¹ We will explain momentarily the mysterious minus sign, which was introduced only for convenience.

Equation (2.3.7) yields two ordinary differential equations, one for $G(t)$ and one for $\phi(x)$:

$$\frac{d^2 \phi}{dx^2} = -\lambda \phi \quad (2.3.8)$$

$$\frac{dG}{dt} = -\lambda k G. \quad (2.3.9)$$

We reiterate that λ is a constant and it is the same constant that appears in both (2.3.8) and (2.3.9). The product solutions, $u(x, t) = \phi(x)G(t)$, must also satisfy the two homogeneous boundary conditions. For example, $u(0, t) = 0$ implies that $\phi(0)G(t) = 0$. There are two possibilities. Either $G(t) \equiv 0$ (the meaning of \equiv is identically zero, for all t) or $\phi(0) = 0$. If $G(t) \equiv 0$, then from (2.3.4), the assumed product solution is identically zero, $u(x, t) \equiv 0$. This is not very interesting. [$u(x, t) \equiv 0$ is called the **trivial solution** since $u(x, t) \equiv 0$ automatically satisfies any homogeneous PDE and any homogeneous BC.] Instead, we look for nontrivial solutions. For nontrivial solutions, we must have

$$\phi(0) = 0. \quad (2.3.10)$$

¹As further explanation for the constant in (2.3.7), let us say the following. Suppose that the left-hand side of (2.3.7) is some function of t , $(1/kG)dG/dt = w(t)$. If we differentiate with respect to x , we get zero: $0 = d/dx(1/\phi d^2 \phi/dx^2)$. Since $1/\phi d^2 \phi/dx^2$ is only a function of x , this implies that $1/\phi d^2 \phi/dx^2$ must be a constant, its derivative equaling zero. In this way (2.3.7) follows.

By applying the other boundary condition, $u(L, t) = 0$, we obtain in a similar way that

$$\phi(L) = 0. \quad (2.3.11)$$

Product solutions, in addition to satisfying two ordinary differential equations, (2.3.8) and (2.3.9), must also satisfy boundary conditions (2.3.10) and (2.3.11).

2.3.3 Time-Dependent Equation

The advantage of the product method is that it transforms a partial differential equation, which we do not know how to solve, into two ordinary differential equations. The boundary conditions impose two conditions on the x -dependent ordinary differential equation (ODE). The time-dependent equation has no additional conditions, just

$$\frac{dG}{dt} = -\lambda k G. \quad (2.3.12)$$

Let us solve (2.3.12) first before we discuss solving the x -dependent ODE with its two homogeneous boundary conditions. Equation (2.3.12) is a first-order linear homogeneous differential equation *with constant coefficients*. We can obtain its general solution quite easily. Nearly all constant-coefficient (linear and homogeneous) ODEs can be solved by seeking exponential solutions, $G = e^{rt}$, where in this case by substitution the characteristic polynomial is $r = -\lambda k$. Therefore, the general solution of (2.3.12) is

$$G(t) = ce^{-\lambda kt}. \quad (2.3.13)$$

We have remembered that for linear homogeneous equations, if $e^{-\lambda kt}$ is a solution, then $ce^{-\lambda kt}$ is a solution (for any arbitrary multiplicative constant c). The time-dependent solution is a simple exponential. Recall that λ is the separation constant, which for the moment is arbitrary. However, eventually we will discover that only certain values of λ are allowable. If $\lambda > 0$, the solution exponentially decays as t increases (since $k > 0$). If $\lambda < 0$, the solution exponentially increases, and if $\lambda = 0$, the solution remains constant in time. Since this is a heat conduction problem and the temperature $u(x, t)$ is proportional to $G(t)$, we do not *expect* the solution to grow exponentially in time. Thus, we expect $\lambda \geq 0$; we have not proved that statement, but *we shouldn't*. Thus, it is rather convenient that we have discovered that we expect $\lambda \geq 0$. In fact, that is why we introduced the expression $-\lambda$ when we separated variables [see (2.3.7)]. If we had introduced μ (instead of $-\lambda$), then our previous arguments would have suggested that $\mu \leq 0$. In summary, when separating variables in (2.3.7), we mentally solve the time-dependent equation and see that $G(t)$ does not exponentially grow only if the separation constant was ≤ 0 . We then introduce $-\lambda$ for convenience, since we would now *expect* $\lambda \geq 0$. We next show how we actually determine all allowable separation constants. We will verify mathematically that $\lambda \geq 0$, as we expect by the physical arguments presented above.

2.3.4 Boundary Value Problem

The x -dependent part of the assumed product solution, $\phi(x)$, satisfies a second-order ODE with two homogeneous boundary conditions:

$$\begin{aligned} \frac{d^2\phi}{dx^2} &= -\lambda\phi \\ \phi(0) &= 0 \\ \phi(L) &= 0. \end{aligned} \quad (2.3.14)$$

We call (2.3.14) a **boundary value problem** for ordinary differential equations. In the usual first course in ordinary differential equations, only initial value problems are specified. For example (think of Newton's law of motion for a particle), we solve second-order differential equations ($m d^2y/dt^2 = F$) subject to two initial conditions [$y(0)$ and $dy/dt(0)$ given] *both at the same time*. Initial value problems are quite nice, as *usually* there *exist unique* solutions to initial value problems. However, (2.3.14) is quite different. It is a boundary value problem, since the two conditions are not given at the same place (e.g., $x = 0$) but at two different places, $x = 0$ and $x = L$. There is no simple theory which guarantees that the solution exists or is unique to this type of problem. In particular, we note that $\phi(x) = 0$ satisfies the ODE and both homogeneous boundary conditions, no matter what the separation constant λ is, even if $\lambda < 0$; it is referred to as the **trivial solution** of the boundary value problem. It corresponds to $u(x, t) \equiv 0$, since $u(x, t) = \phi(x)G(t)$. If solutions of (2.3.14) had been unique, then $\phi(x) \equiv 0$ would be the only solution; we would not be able to obtain nontrivial solutions of a linear homogeneous PDE by the product (separation of variables) method. Fortunately, there are other solutions of (2.3.14). However, there do not exist nontrivial solutions of (2.3.14) for all values of λ . Instead, we will show that there are certain special values of λ , called **eigenvalues**² of the boundary value problem (2.3.14), for which there are nontrivial solutions, $\phi(x)$. A nontrivial $\phi(x)$, which exists only for certain values of λ , is called an **eigenfunction** corresponding to the eigenvalue λ .

Let us try to determine the eigenvalues λ . In other words, for what values of λ are there nontrivial solutions of (2.3.14)? We solve (2.3.14) directly. The second-order ODE is linear and homogeneous with constant coefficients: two independent solutions are usually obtained in the form of exponentials, $\phi = e^{rx}$. Substituting this exponential into the differential equation yields the characteristic polynomial $r^2 = -\lambda$. The solutions corresponding to the two roots have significantly different properties depending on the value of λ . There are four cases:

1. $\lambda > 0$, in which the two roots are purely imaginary and are complex conjugates of each other, $r = \pm i\sqrt{\lambda}$.
2. $\lambda = 0$, in which the two roots coalesce and are equal, $r = 0, 0$.

²The word *eigenvalue* comes from the German word *eigenwert*, meaning characteristic value.

3. $\lambda < 0$, in which the two roots are real and unequal, $r = \pm\sqrt{-\lambda}$, one positive and one negative. (Note that in this case $-\lambda$ is positive, so that the square root operation is well defined.)
4. λ itself complex.

We will ignore the last case (as most of you would have done anyway) since we will later (Chapter 5) prove that λ is real in order for a nontrivial solution of the boundary value problem (2.3.14) to exist. From the time-dependent solution, using physical reasoning, we expect that $\lambda \geq 0$; perhaps then it will be unnecessary to analyze case 3. However, we will demonstrate a mathematical reason for the omission of this case.

Eigenvalues and eigenfunctions ($\lambda > 0$). Let us first consider the case in which $\lambda > 0$. The boundary value problem is

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad (2.3.15)$$

$$\phi(0) = 0 \quad (2.3.16)$$

$$\phi(L) = 0 \quad (2.3.17)$$

If $\lambda > 0$, exponential solutions have imaginary exponents, $e^{\pm i\sqrt{\lambda}x}$. In this case, the solutions oscillate. If we desire real independent solutions, the choices $\cos\sqrt{\lambda}x$ and $\sin\sqrt{\lambda}x$ are usually made ($\cos\sqrt{\lambda}x$ and $\sin\sqrt{\lambda}x$ are each linear combinations of $e^{\pm i\sqrt{\lambda}x}$). Thus, the general solution of (2.3.15) is

$$\phi = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x, \quad (2.3.18)$$

an arbitrary linear combination of two independent solutions. (The linear combination may be chosen from any two independent solutions.) $\cos\sqrt{\lambda}x$ and $\sin\sqrt{\lambda}x$ are usually the most convenient, but $e^{i\sqrt{\lambda}x}$ and $e^{-i\sqrt{\lambda}x}$ can be used. In some examples, other independent solutions are chosen. For example, Exercise 2.3.2(f) illustrates the advantage of sometimes choosing $\cos\sqrt{\lambda}(x-a)$ and $\sin\sqrt{\lambda}(x-a)$ as independent solutions.

We now apply the boundary conditions. $\phi(0) = 0$ implies that

$$0 = c_1.$$

The cosine term vanishes, since the solution must be zero at $x = 0$. Thus, $\phi(x) = c_2 \sin\sqrt{\lambda}x$. Only the boundary condition at $x = L$ has not been satisfied. $\phi(L) = 0$ implies that

$$0 = c_2 \sin\sqrt{\lambda}L.$$

Either $c_2 = 0$ or $\sin\sqrt{\lambda}L = 0$. If $c_2 = 0$, then $\phi(x) \equiv 0$ since we already determined that $c_1 = 0$. This is the trivial solution, and we are searching for those values of λ that have nontrivial solutions. The eigenvalues λ must satisfy

$$\sin\sqrt{\lambda}L = 0. \quad (2.3.19)$$

$\sqrt{\lambda}L$ must be a zero of the sine function. A sketch of $\sin z$ (see Fig. 2.3.1) or our knowledge of the sine function shows that $\sqrt{\lambda}L = n\pi$. $\sqrt{\lambda}L$ must equal an integral multiple of π , where n is a positive integer since $\sqrt{\lambda} > 0$ ($n = 0$ is not appropriate since we assumed that $\lambda > 0$ in this derivation). The **eigenvalues** λ are

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \dots \quad (2.3.20)$$

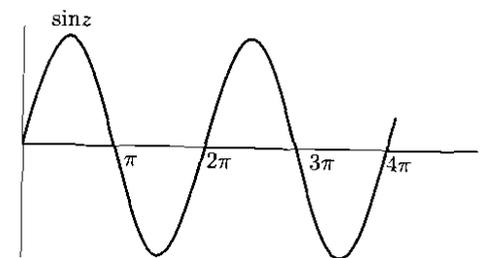


Figure 2.3.1: Zeros of $\sin z$.

The eigenfunction corresponding to the eigenvalue $\lambda = (n\pi/L)^2$ is

$$\phi(x) = c_2 \sin\sqrt{\lambda}x = c_2 \sin\frac{n\pi x}{L}, \quad (2.3.21)$$

where c_2 is an arbitrary multiplicative constant. Often we pick a convenient value for c_2 , for example $c_2 = 1$. We should remember, though, that any specific eigenfunction can always be multiplied by an arbitrary constant, since the PDE and BCs are linear and homogeneous.

Eigenvalue ($\lambda = 0$). Now we will determine if $\lambda = 0$ is an eigenvalue for (2.3.15) subject to the boundary conditions (2.3.16), (2.3.17). $\lambda = 0$ is a special case. If $\lambda = 0$, (2.3.15) implies that

$$\phi = c_1 + c_2x,$$

corresponding to the double-zero roots, $r = 0, 0$ of the characteristic polynomial.³ To determine whether $\lambda = 0$ is an eigenvalue, the homogeneous boundary conditions must be applied. $\phi(0) = 0$ implies that $0 = c_1$, and thus $\phi = c_2x$. In addition, $\phi(L) = 0$ implies that $0 = c_2L$. Since the length L of the rod is positive ($\neq 0$), $c_2 = 0$ and thus $\phi(x) \equiv 0$. This is the trivial solution, so we say that $\lambda = 0$ is *not* an eigenvalue, for *this* problem [(2.3.15), (2.3.16), (2.3.17)]. Be wary, though; $\lambda = 0$ is an eigenvalue for other problems and should be looked at individually for any new problem you may encounter.

³Please do *not* say that $\phi = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$ is the general solution for $\lambda = 0$. If you do that, you find for $\lambda = 0$ that the general solution is an arbitrary constant. Although an arbitrary constant solves (2.3.15) when $\lambda = 0$, (2.3.15) is still a linear second-order differential equation; its general solution must be a linear combination of *two* independent solutions. It is possible to choose $\sin\sqrt{\lambda}x/\sqrt{\lambda}$ as a second independent solution so that as $\lambda \rightarrow 0$ it agrees with the solution x . However, this involves too much work. It is better just to consider $\lambda = 0$ as a separate case.

Eigenvalues ($\lambda < 0$). Are there any negative eigenvalues? If $\lambda < 0$, the solution of

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad (2.3.22)$$

is not difficult, but you may have to be careful. The roots of the characteristic polynomial are $r = \pm\sqrt{-\lambda}$, so solutions are $e^{\sqrt{-\lambda}x}$ and $e^{-\sqrt{-\lambda}x}$. If you do not like the notation $\sqrt{-\lambda}$, you may prefer what is equivalent (if $\lambda < 0$), namely $\sqrt{|\lambda|}$. However, $\sqrt{|\lambda|} \neq \sqrt{\lambda}$ since $\lambda < 0$. It is convenient to let

$$\lambda = -s,$$

in the case in which $\lambda < 0$. Then $s > 0$, and the differential equation (2.3.22) becomes

$$\frac{d^2\phi}{dx^2} = s\phi. \quad (2.3.23)$$

Two independent solutions are $e^{+\sqrt{s}x}$ and $e^{-\sqrt{s}x}$, since $s > 0$. The general solution is

$$\phi = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}. \quad (2.3.24)$$

Frequently, we use the hyperbolic functions instead. As a review, the definitions of the hyperbolic functions are

$$\cosh z \equiv \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z \equiv \frac{e^z - e^{-z}}{2},$$

simple linear combinations of exponentials. These are sketched in Fig. 2.3.2. Note that $\sinh 0 = 0$ and $\cosh 0 = 1$ (the results analogous to those for trigonometric functions). Also note that $d/dz \cosh z = \sinh z$ and $d/dz \sinh z = \cosh z$, quite similar to trigonometric functions, but easier to remember because of the lack of the annoying appearance of any minus signs in the differentiation formulas. If hyperbolic functions are used instead of exponentials, the general solution of (2.3.23) can be written as

$$\phi = c_3 \cosh \sqrt{s}x + c_4 \sinh \sqrt{s}x, \quad (2.3.25)$$

a form equivalent to (2.3.24). To determine if there are any negative eigenvalues ($\lambda < 0$, but $s > 0$ since $\lambda = -s$), we again apply the boundary conditions. Either form (2.3.24) or (2.3.25) can be used; the same answer is obtained either way. From (2.3.25), $\phi(0) = 0$ implies that $0 = c_3$, and hence $\phi = c_4 \sinh \sqrt{s}x$. The other boundary condition, $\phi(L) = 0$, implies that $c_4 \sinh \sqrt{s}L = 0$. Since $\sqrt{s}L > 0$ and since \sinh is never zero for a positive argument (see Fig. 2.3.2), it follows that $c_4 = 0$. Thus, $\phi(x) \equiv 0$. The only solution of (2.3.23) for $\lambda < 0$ that solves the homogeneous boundary conditions is the trivial solution. Thus, there are no negative eigenvalues. For this example, the existence of negative eigenvalues would have corresponded to exponential growth in time. We did not expect such solutions on physical grounds, and here we have verified mathematically in an explicit manner that there cannot be any negative eigenvalues for this problem. In some other problems there can be negative eigenvalues. Later (Sec. 5.3) we will formulate a theory, involving the Rayleigh quotient, in which we will know before we start many problems that there

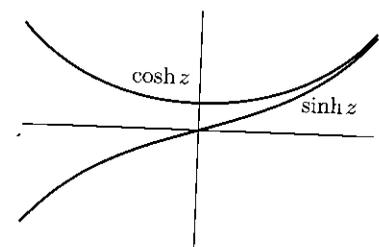


Figure 2.3.2: Hyperbolic functions.

cannot be negative eigenvalues. This will at times eliminate calculations such as the ones just performed.

Eigenfunctions — summary. We summarize our results for the boundary value problem resulting from separation of variables:

$$\begin{aligned} \frac{d^2\phi}{dx^2} + \lambda\phi &= 0 \\ \phi(0) &= 0 \\ \phi(L) &= 0. \end{aligned}$$

This boundary value problem will arise many times in the text. It is helpful to nearly memorize the result that the eigenvalues λ are all positive (not zero or negative),

$$\lambda = \left(\frac{n\pi}{L}\right)^2,$$

where n is any positive integer, $n = 1, 2, 3, \dots$, and the corresponding eigenfunctions are

$$\phi(x) = \sin \frac{n\pi x}{L}.$$

If we introduce the notation λ_1 for the first (or lowest) eigenvalue, λ_2 for the next, and so on, we see that $\lambda_n = (n\pi/L)^2$, $n = 1, 2, \dots$. The corresponding eigenfunctions are sometimes denoted $\phi_n(x)$, the first few of which are sketched in Fig. 2.3.3. All eigenfunctions are (of course) zero at both $x = 0$ and $x = L$. Notice that $\phi_1(x) = \sin \pi x/L$ has no zeros for $0 < x < L$, and $\phi_2(x) = \sin 2\pi x/L$ has one zero for $0 < x < L$. In fact, $\phi_n(x) = \sin n\pi x/L$ has $n - 1$ zeros for $0 < x < L$. We will claim later (see Sec 5.3) that, remarkably, this is a general property of eigenfunctions.

Spring-mass analog. We have obtained solutions of $d^2\phi/dx^2 = -\lambda\phi$. Here we present the analog of this to a spring-mass system, which some of you may find helpful. A spring-mass system subject to Hooke's law satisfies $md^2y/dt^2 =$

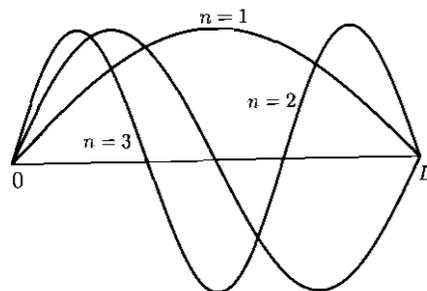


Figure 2.3.3: Eigenfunctions $\sin n\pi x/L$ and their zeros.

$-ky$, where $k > 0$ is the spring constant. Thus, if $\lambda > 0$, the ODE (2.3.15) may be thought of as a spring-mass system with a restoring force. Thus, if $\lambda > 0$ the solution should oscillate. It should not be surprising that the BCs (2.3.16, 2.3.17) can be satisfied for $\lambda > 0$; a nontrivial solution of the ODE, which is zero at $x = 0$, has a chance of being zero again at $x = L$ since there is a restoring force and the solution of the ODE oscillates. We have shown that this can happen for specific values of $\lambda > 0$. However, if $\lambda < 0$, then the force is not restoring. It would seem less likely that a nontrivial solution which is zero at $x = 0$ could possibly be zero again at $x = L$. We must not always trust our intuition entirely, so we have verified these facts mathematically.

2.3.5 Product Solutions and the Principle of Superposition

In summary, we obtained product solutions of the heat equation, $\partial u/\partial t = k\partial^2 u/\partial x^2$, satisfying the specific homogeneous boundary conditions $u(0, t) = 0$ and $u(L, t) = 0$ only corresponding to $\lambda > 0$. These solutions, $u(x, t) = \phi(x)G(t)$, have $G(t) = ce^{-\lambda kt}$ and $\phi(x) = c_2 \sin \sqrt{\lambda}x$, where we determined from the boundary conditions [$\phi(0) = 0$ and $\phi(L) = 0$] the allowable values of the separation constant λ , $\lambda = (n\pi/L)^2$. Here n is a positive integer. Thus, product solutions of the heat equation are

$$u(x, t) = B \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}, \quad n = 1, 2, \dots, \quad (2.3.26)$$

where B is an arbitrary constant ($B = cc_2$). This is a different solution for each n . Note that as t increases, these special solutions exponentially decay, in particular, for these solutions, $\lim_{t \rightarrow \infty} u(x, t) = 0$. In addition, $u(x, t)$ satisfies a special initial condition, $u(x, 0) = B \sin n\pi x/L$.

Initial value problems. We can use the simple product solutions (2.3.26), to satisfy an initial value problem if the initial condition happens to be just right,

For example, suppose that we wish to solve the following initial value problem:

$$\begin{aligned} \text{PDE: } \quad \frac{\partial u}{\partial t} &= k \frac{\partial^2 u}{\partial x^2} \\ \text{BC: } \quad u(0, t) &= 0 \\ &u(L, t) = 0 \\ \text{IC: } \quad u(x, 0) &= 4 \sin \frac{3\pi x}{L} \end{aligned}$$

Our product solution $u(x, t) = B \sin n\pi x/L \cdot e^{-k(n\pi/L)^2 t}$ satisfies the initial condition $u(x, 0) = B \sin n\pi x/L$. Thus, by picking $n = 3$ and $B = 4$, we will have satisfied the initial condition. Our solution of this example is thus

$$u(x, t) = 4 \sin \frac{3\pi x}{L} e^{-k(3\pi/L)^2 t}.$$

It can be proved that this physical problem (as well as most we consider) has a unique solution. Thus, it does not matter what procedure we used to obtain the solution.

Principle of superposition. The product solutions appear to be very special, since they may be used directly only if the initial condition happens to be of the appropriate form. However, we wish to show that these solutions are useful in many other situations, in fact, in all situations. Consider the same PDE and BCs, but instead subject to the initial condition

$$u(x, 0) = 4 \sin \frac{3\pi x}{L} + 7 \sin \frac{8\pi x}{L}.$$

The solution of this problem can be obtained by adding together two simpler solutions obtained by the product method:

$$u(x, t) = 4 \sin \frac{3\pi x}{L} e^{-k(3\pi/L)^2 t} + 7 \sin \frac{8\pi x}{L} e^{-k(8\pi/L)^2 t}.$$

We immediately see that this solves the initial condition (substitute $t = 0$) as well as the boundary conditions (substitute $x = 0$ and $x = L$). Only slightly more work shows that the partial differential equation has been satisfied. This is an illustration of the principle of superposition.

Superposition (extended). The principle of superposition can be extended to show that if $u_1, u_2, u_3, \dots, u_M$ are solutions of a linear homogeneous problem, then any linear combination of these is also a solution, $c_1 u_1 + c_2 u_2 + c_3 u_3 + \dots + c_M u_M = \sum_{n=1}^M c_n u_n$, where c_n are arbitrary constants. Since we know from the method of separation of variables that $\sin n\pi x/L \cdot e^{-k(n\pi/L)^2 t}$ is a solution of the heat equation (solving zero boundary conditions) for all positive n , it follows that any linear combination of these solutions is also a solution of the linear homogeneous heat equation. Thus,

$$u(x, t) = \sum_{n=1}^M B_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t} \quad (2.3.27)$$

solves the heat equation (with zero boundary conditions) for any finite M . We have added solutions to the heat equation, keeping in mind that the “amplitude” B could be different for each solution, yielding the subscript B_n . Equation (2.3.27) shows that we can solve the heat equation if initially

$$u(x, 0) = f(x) = \sum_{n=1}^M B_n \sin \frac{n\pi x}{L}, \quad (2.3.28)$$

that is, if the initial condition equals a finite sum of the appropriate sine functions. What should we do in the usual situation in which $f(x)$ is *not* a finite linear combination of the appropriate sine functions? We claim that the theory of Fourier series (to be described with considerable detail in Chapter 3) states that:

1. Any function $f(x)$ (with certain very reasonable restrictions, to be discussed later) can be approximated (in some sense) by a finite linear combination of $\sin n\pi x/L$.
2. The approximation may not be very good for small M , but gets to be a better and better approximation as M is increased (see Sec. 5.10).
3. Furthermore, if we consider the limit as $m \rightarrow \infty$, then not only is (2.3.28) the best approximation to $f(x)$ using combinations of the eigenfunctions, but (again in some sense) the resulting infinite series will converge to $f(x)$ [with some restrictions on $f(x)$, to be discussed].

We thus claim (and clarify and make precise in Chapter 3) that “any” initial condition $f(x)$ can be written as an infinite linear combination of $\sin n\pi x/L$, known as a type of **Fourier series**:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}. \quad (2.3.29)$$

What is more important is that we also claim that the corresponding infinite series is the solution of our heat conduction problem:

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}. \quad (2.3.30)$$

Analyzing infinite series such as (2.3.29) and (2.3.30) is not easy. We must discuss the convergence of these series as well as briefly discuss the validity of an infinite series solution of our entire problem. For the moment, let us ignore these somewhat theoretical issues and concentrate on the construction of these infinite series solutions.

2.3.6 Orthogonality of Sines

One very important practical point has been neglected. Equation (2.3.30) is our solution with the coefficients B_n satisfying (2.3.29) (from the initial conditions), but how do we determine the coefficients B_n ? We assume it is possible that

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad (2.3.31)$$

where this is to hold over the region of the one-dimensional rod, $0 \leq x \leq L$. We will assume that standard mathematical operations are also valid for infinite series. Equation (2.3.31) represents one equation in an infinite number of unknowns, but it should be valid at every value of x . If we substitute a thousand different values of x into (2.3.31), each of the thousand equations would hold, but there would still be an infinite number of unknowns. This is not an efficient way to determine the B_n . Instead, we frequently will employ an extremely important technique based on noticing (perhaps from a table of integrals) that the eigenfunctions $\sin n\pi x/L$ satisfy the following integral property:

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \begin{cases} 0 & m \neq n \\ L/2 & m = n, \end{cases} \quad (2.3.32)$$

where m and n are positive integers.

To use these conditions, (2.3.32), to determine B_n , we multiply both sides of (2.3.31) by $\sin m\pi x/L$ (for any fixed integer m , independent of the “dummy” index n):

$$f(x) \sin \frac{m\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}. \quad (2.3.33)$$

Next we integrate (2.3.33) from $x = 0$ to $x = L$:

$$\int_0^L f(x) \sin \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx. \quad (2.3.34)$$

For finite series, the integral of a sum of terms equals the sum of the integrals. We assume that this is valid for this infinite series. Now we evaluate the infinite sum. From the integral property (2.3.32), we see that each term of the sum is zero whenever $n \neq m$. In summing over n , eventually n equals m . It is only for that one value of n , i.e., $n = m$, that there is a contribution to the infinite sum. The only term that appears on the right-hand side of (2.3.34) occurs when n is replaced by m :

$$\int_0^L f(x) \sin \frac{m\pi x}{L} dx = B_m \int_0^L \sin^2 \frac{m\pi x}{L} dx.$$

Since the integral on the right equals $L/2$, we can solve for B_m :

$$B_m = \frac{\int_0^L f(x) \sin \frac{m\pi x}{L} dx}{\int_0^L \sin^2 \frac{m\pi x}{L} dx} = \frac{2}{L} \int_0^L f(x) \sin \frac{m\pi x}{L} dx. \quad (2.3.35)$$

This result is very important and so is the method by which it was obtained. Try to learn both. The integral in (2.3.35) is considered to be known since $f(x)$ is the given initial condition. The integral cannot usually be evaluated, in which case numerical integrations (on a computer) may need to be performed to get explicit numbers for B_m , $m = 1, 2, 3, \dots$

You will find that the formula (2.3.32), $\int_0^L \sin^2 n\pi x/L dx = L/2$, is quite useful in many different circumstances, including applications having nothing to do with the material of this text. One reason for its applicability is that there are many periodic phenomena in nature ($\sin \omega t$), and usually energy or power is proportional to the square ($\sin^2 \omega t$). The average energy is then proportional to $\int_0^{2\pi/\omega} \sin^2 \omega t dt$ divided by the period $2\pi/\omega$. It is worthwhile to memorize that the average over a full period of sine or cosine squared is $\frac{1}{2}$. Thus, *the integral over any number of complete periods of the square of a sine or cosine is one-half the length of the interval*. In this way $\int_0^L \sin^2 n\pi x/L dx = L/2$, since the interval 0 to L is either a complete or a half period of $\sin n\pi x/L$.

Orthogonality. Whenever $\int_0^L A(x)B(x) dx = 0$ we say that the functions $A(x)$ and $B(x)$ are **orthogonal** over the interval $0 \leq x \leq L$. We borrow the terminology "orthogonal" from perpendicular vectors because $\int_0^L A(x)B(x) dx = 0$ is analogous to a zero dot product, as is explained further in the appendix to this section. A set of functions each member of which is orthogonal to every *other* member is called an **orthogonal set of functions**. An example is that of the functions $\sin n\pi x/L$, the eigenfunctions of the boundary value problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

They are mutually orthogonal because of (2.3.32). Therefore, we call (2.3.32) an orthogonality condition.

In fact, we will discover that for most other boundary value problems, the eigenfunctions will form an orthogonal set of functions (with certain modifications discussed in Chapter 5 with respect to Sturm-Liouville eigenvalue problems).

2.3.7 Formulation, Solution, and Interpretation of an Example

As an example, let us analyze our solution in the case in which the initial temperature is constant, 100°C . This corresponds to a physical problem that is easy to reproduce in the laboratory. Take a one-dimensional rod and place the entire rod

in a large tub of boiling water (100°C). Let it sit there for a long time. After a while (we expect) the rod will be at 100°C throughout. Now insulate the lateral sides (if that had not been done earlier) and suddenly (at $t = 0$) immerse the two ends in large well-stirred baths of ice water, 0°C . The mathematical problem is

$$\text{PDE: } \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad t > 0, \quad 0 < x < L \quad (2.3.36)$$

$$\text{BC: } \begin{aligned} u(0, t) &= 0 \\ u(L, t) &= 0 \end{aligned} \quad t > 0 \quad (2.3.37)$$

$$\text{IC: } u(x, 0) = 100 \quad 0 < x < L. \quad (2.3.38)$$

According to (2.3.30) and (2.3.35), the solution is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}, \quad (2.3.39)$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \quad (2.3.40)$$

and $f(x) = 100$. Recall that the coefficient B_n was determined by having (2.3.39) satisfy the initial condition,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}. \quad (2.3.41)$$

We calculate the coefficients B_n from (2.3.40):

$$\begin{aligned} B_n &= \frac{2}{L} \int_0^L 100 \sin \frac{n\pi x}{L} dx = \frac{200}{L} \left(-\frac{L}{n\pi} \cos \frac{n\pi x}{L} \right) \Big|_0^L \\ &= \frac{200}{n\pi} (1 - \cos n\pi) = \begin{cases} 0 & n \text{ even} \\ \frac{400}{n\pi} & n \text{ odd} \end{cases} \end{aligned} \quad (2.3.42)$$

since $\cos n\pi = (-1)^n$ which equals 1 for n even and -1 for n odd. The series (2.3.41) will be studied further in Chapter 3. In particular, we must explain the intriguing situation that the initial temperature equals 100 everywhere, but the series (2.3.41) equals 0 at $x = 0$ and $x = L$ (due to the boundary conditions).

Approximations to the initial value problem. We have now obtained the solution to the initial value problem (2.3.36)-(2.3.38) for the heat equation with zero boundary conditions ($x = 0$ and $x = L$) and initial temperature distribution equaling 100. The solution is (2.3.39), with B_n given by (2.3.42). The solution is quite complicated, involving an infinite series. What can we say about it? First, we notice that $\lim_{t \rightarrow \infty} u(x, t) = 0$. The temperature distribution approaches a steady state, $u(x, t) = 0$. This is not surprising physically since both ends are at

0° ; we expect all the initial heat energy contained in the rod to flow out the ends. The equilibrium problem, $d^2u/dx^2 = 0$ with $u(0) = 0$ and $u(L) = 0$, has a unique solution, $u \equiv 0$, agreeing with the limit as t tends to infinity of the time-dependent problem.

One question of importance that we can answer is the manner in which the solution approaches steady state. If t is large, what is the approximate temperature distribution, and how does it differ from the steady state 0° ? We note that each term in (2.3.39) decays at a different rate. The more oscillations in space, the faster the decay. If t is such that $kt(\pi/L)^2$ is large, then each succeeding term is much smaller than the first. We can then approximate the infinite series by only the first term:

$$u(x, t) \approx \frac{400}{\pi} \sin \frac{\pi x}{L} e^{-k(\pi/L)^2 t}. \quad (2.3.43)$$

The larger t is, the better this is as an approximation. Even if $kt(\pi/L)^2 = \frac{1}{2}$ this is not a bad approximation since

$$\frac{e^{-k(3\pi/L)^2 t}}{e^{-k(\pi/L)^2 t}} = e^{-8(\pi/L)^2 kt} = e^{-4} = 0.018 \dots$$

Thus, if $kt(\pi/L)^2 \geq \frac{1}{2}$, we can use the simple approximation. We see that for these times the spatial dependence of the temperature is just the simple rise and fall of $\sin \pi x/L$, as illustrated in Fig. 2.3.4. The peak amplitude, occurring in the middle $x = L/2$, decays exponentially in time. For $kt(\pi/L)^2$ less than $\frac{1}{2}$, the spatial dependence cannot be approximated by one simple sinusoidal function; more terms are necessary in the series. The solution can be easily computed, using a finite number of terms. In some cases many terms may be necessary, and there would be better ways to calculate $u(x, t)$.

2.3.8 Summary

Let us summarize the method of separation of variables as it appears for the one example:

$$\text{PDE: } \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$$\text{BC1: } u(0, t) = 0$$

$$\text{BC2: } u(L, t) = 0$$

$$\text{IC: } u(x, 0) = f(x).$$

1. Make sure that you have a linear and homogeneous PDE with linear and homogeneous BC.
2. Temporarily ignore the nonzero IC.
3. Separate variables (determine differential equations implied by the assumption of product solutions) and introduce a separation constant.

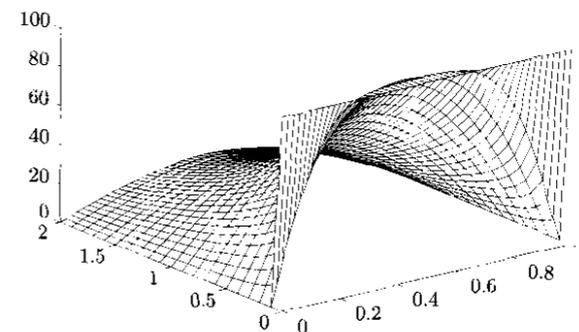


Figure 2.3.4: Time dependence of temperature $u(x, t)$.

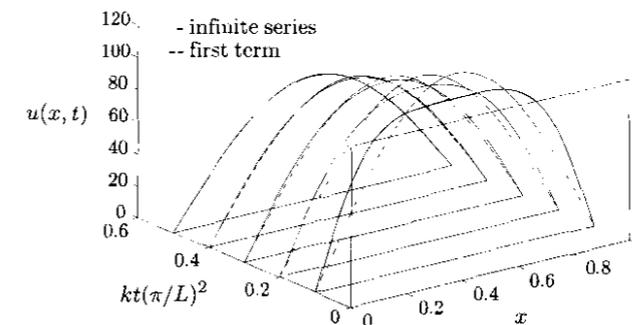


Figure 2.3.5: Time dependence of temperature (using the infinite series) compared to the first term. Note the first term is a good approximation if the time is not too small.

4. Determine separation constants as the eigenvalues of a boundary value problem.
5. Solve other differential equations. Record all product solutions of the PDE obtainable by this method.
6. Apply the principle of superposition (for a linear combination of all product solutions).
7. Attempt to satisfy the initial condition.
8. Determine coefficients using the orthogonality of the eigenfunctions.

These steps should be *understood*, not memorized. It is important to note that

1. The principle of superposition applies to solutions of the PDE (do not add up solutions of various different ordinary differential equations).
2. Do not apply the initial condition $u(x, 0) = f(x)$ until *after* the principle of superposition.

EXERCISES 2.3

2.3.1. For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

$$*(a) \quad \frac{\partial u}{\partial t} = \frac{k}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \qquad (b) \quad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - v_0 \frac{\partial u}{\partial x}$$

$$*(c) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (d) \quad \frac{\partial u}{\partial t} = \frac{k}{r} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right)$$

$$*(e) \quad \frac{\partial u}{\partial t} = k \frac{\partial^4 u}{\partial x^4} \qquad *(f) \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

2.3.2. Consider the differential equation

$$\frac{d^2 \phi}{dx^2} + \lambda \phi = 0.$$

Determine the eigenvalues λ (and corresponding eigenfunctions), if ϕ satisfies the following boundary conditions. Analyze three cases ($\lambda > 0$, $\lambda = 0$, $\lambda < 0$). You may assume that the eigenvalues are real.

- (a) $\phi(0) = 0$ and $\phi(\pi) = 0$
 *(b) $\phi(0) = 0$ and $\phi(1) = 0$
 (c) $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$ (If necessary, see Sec. 2.4.1.)
 *(d) $\phi(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$
 (e) $\frac{d\phi}{dx}(0) = 0$ and $\phi(L) = 0$
 *(f) $\phi(a) = 0$ and $\phi(b) = 0$ (You may assume that $\lambda > 0$.)
 (g) $\phi(0) = 0$ and $\frac{d\phi}{dx}(L) + \phi(L) = 0$ (If necessary, see Sec. 5.8.)

2.3.3. Consider the heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions

$$u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0.$$

Solve the initial value problem if the temperature is initially

- (a) $u(x, 0) = 6 \sin \frac{9\pi x}{L}$ (b) $u(x, 0) = 3 \sin \frac{\pi x}{L} - \sin \frac{3\pi x}{L}$
 *(c) $u(x, 0) = 2 \cos \frac{3\pi x}{L}$ (d) $u(x, 0) = \begin{cases} 1 & 0 < x \leq L/2 \\ 2 & L/2 < x < L \end{cases}$

[Your answer in part (c) may involve certain integrals that do not need to be evaluated.]

2.3.4. Consider

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},$$

subject to $u(0, t) = 0$, $u(L, t) = 0$, and $u(x, 0) = f(x)$.

- *(a) What is the total heat energy in the rod as a function of time?
 (b) What is the flow of heat energy out of the rod at $x = 0$? at $x = L$?
 *(c) What relationship should exist between parts (a) and (b)?

2.3.5. Evaluate (be careful if $n = m$)

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx \quad \text{for } n > 0, m > 0.$$

Use the trigonometric identity

$$\sin a \sin b = \frac{1}{2} [\cos(a - b) - \cos(a + b)].$$

*2.3.6. Evaluate

$$\int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx \quad \text{for } n \geq 0, m \geq 0.$$

Use the trigonometric identity

$$\cos a \cos b = \frac{1}{2} [\cos(a + b) + \cos(a - b)].$$

(Be careful if $a - b = 0$ or $a + b = 0$.)

2.3.7. Consider the following boundary value problem (if necessary, see Sec. 2.4.1):

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \frac{\partial u}{\partial x}(0, t) = 0, \frac{\partial u}{\partial x}(L, t) = 0, \quad \text{and} \quad u(x, 0) = f(x)$$

- (a) Give a one-sentence physical interpretation of this problem.
 (b) Solve by the method of separation of variables. First show that there are no separated solutions which exponentially grow in time. [Hint: The answer is

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n k t} \cos \frac{n\pi x}{L}.$$

What is λ_n ?

(c) Show that the initial condition, $u(x, 0) = f(x)$, is satisfied if

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}.$$

(d) Using Exercise 2.3.6, solve for A_0 and A_n ($n \geq 1$).

(e) What happens to the temperature distribution as $t \rightarrow \infty$? Show that it approaches the steady-state temperature distribution (see Sec. 1.4).

*2.3.8. Consider

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - \alpha u.$$

This corresponds to a one-dimensional rod either with heat loss through the lateral sides with outside temperature 0° ($\alpha > 0$, see Exercise 1.2.4) or with insulated lateral sides with a heat source proportional to the temperature. Suppose that the boundary conditions are

$$u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0.$$

(a) What are the possible equilibrium temperature distributions if $\alpha > 0$?

(b) Solve the time-dependent problem [$u(x, 0) = f(x)$] if $\alpha > 0$. Analyze the temperature for large time ($t \rightarrow \infty$) and compare to part (a).

*2.3.9. Redo Exercise 2.3.8 if $\alpha < 0$. [Be especially careful if $-\alpha/k = (n\pi/L)^2$.]

2.3.10. For two- and three-dimensional vectors, fundamental property of dot products, $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta$, implies that

$$|\mathbf{A} \cdot \mathbf{B}| \leq |\mathbf{A}||\mathbf{B}|. \quad (2.3.44)$$

In this exercise we generalize this to n -dimensional vectors and functions, in which case (2.3.44) is known as **Schwarz's inequality**. [The names of Cauchy and Buniakovsky are also associated with (2.3.44).]

(a) Show that $|\mathbf{A} - \gamma \mathbf{B}|^2 > 0$ implies (2.3.44), where $\gamma = \mathbf{A} \cdot \mathbf{B} / \mathbf{B} \cdot \mathbf{B}$.

(b) Express the inequality using both

$$\mathbf{A} \cdot \mathbf{B} = \sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_n c_n \frac{b_n}{c_n}.$$

*(c) Generalize (2.3.44) to functions. [Hint: Let $\mathbf{A} \cdot \mathbf{B}$ mean $\int_0^L A(x)B(x) dx$.]

2.3.11. Solve Laplace's equation inside a rectangle:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the boundary conditions

$$\begin{aligned} u(0, y) &= g(y) & u(x, 0) &= 0 \\ u(L, y) &= 0 & u(x, H) &= 0. \end{aligned}$$

(Hint: If necessary, see Sec. 2.5.1.)

Appendix to 2.3: Orthogonality of Functions

Two vectors \mathbf{A} and \mathbf{B} are orthogonal if $\mathbf{A} \cdot \mathbf{B} = 0$. In component form: $\mathbf{A} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ and $\mathbf{B} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$; \mathbf{A} and \mathbf{B} are orthogonal if $\sum_i a_i b_i = 0$. A function $A(x)$ can be thought of as a vector. If only three values of x are important, x_1, x_2 , and x_3 , then the components of the function $A(x)$ (thought of as a vector) are $A(x_1) \equiv a_1, A(x_2) \equiv a_2$, and $A(x_3) \equiv a_3$. The function $A(x)$ is orthogonal to the function $B(x)$ (by definition) if $\sum_i a_i b_i = 0$. However, in our problems, all values of x between 0 and L are important. The function $A(x)$ can be thought of as an infinite-dimensional vector, whose components are $A(x_i)$ for all x_i on some interval. In this manner the function $A(x)$ would be said to be orthogonal to $B(x)$ if $\sum_i A(x_i)B(x_i) = 0$, where the summation was to include all points between 0 and L . It is thus natural to define the function $A(x)$ to be orthogonal to $B(x)$ if $\int_0^L A(x)B(x) dx = 0$. The integral replaces the vector dot product; both are examples of "inner products."

In vectors, we have the three mutually perpendicular (orthogonal) unit vectors \hat{i}, \hat{j} , and \hat{k} , known as the standard basis vectors. In component form

$$\mathbf{A} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}.$$

a_1 is the projection of \mathbf{A} in the \hat{i} direction, and so on. Sometimes we wish to represent \mathbf{A} in terms of other mutually orthogonal vectors (which may not be unit vectors) \mathbf{u}, \mathbf{v} , and \mathbf{w} , called an *orthogonal set of vectors*. Then

$$\mathbf{A} = \alpha_u \mathbf{u} + \alpha_v \mathbf{v} + \alpha_w \mathbf{w}.$$

To determine the coordinates $\alpha_u, \alpha_v, \alpha_w$ with respect to this orthogonal set, \mathbf{u}, \mathbf{v} , and \mathbf{w} , we can form certain dot products. For example,

$$\mathbf{A} \cdot \mathbf{u} = \alpha_u \mathbf{u} \cdot \mathbf{u} + \alpha_v \mathbf{v} \cdot \mathbf{u} + \alpha_w \mathbf{w} \cdot \mathbf{u}.$$

Note that $\mathbf{v} \cdot \mathbf{u} = 0$ and $\mathbf{w} \cdot \mathbf{u} = 0$, since we assumed that this new set was mutually orthogonal. Thus, we can easily solve for the coordinate α_u , of \mathbf{A} in the \mathbf{u} direction,

$$\alpha_u = \frac{\mathbf{A} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}. \quad (2.3.45)$$

($\alpha_u \mathbf{u}$ is the vector projection of \mathbf{A} in the \mathbf{u} direction.)

For functions, we can do a similar thing. If $f(x)$ can be represented by a linear combination of the orthogonal set, $\sin n\pi x/L$, then

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L},$$

where the B_n may be interpreted as the coordinates of $f(x)$ with respect to the "direction" (or basis vector) $\sin n\pi x/L$. To determine these coordinates we take the inner product with an arbitrary basis function (vector) $\sin n\pi x/L$, where the inner product of two functions is the integral of their product. Thus, as before,

$$\int_0^L f(x) \sin \frac{n\pi x}{L} dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx.$$

Since $\sin n\pi x/L$ is an orthogonal set of functions, $\int_0^L \sin n\pi x/L \sin m\pi x/L dx = 0$ for $n \neq m$. Hence, we solve for the coordinate (coefficient) B_n :

$$B_n = \frac{\int_0^L f(x) \sin n\pi x/L dx}{\int_0^L \sin^2 n\pi x/L dx}. \quad (2.3.46)$$

This is seen to be the same idea as the projection formula (2.3.45). Our standard formula (2.3.33), $\int_0^L \sin^2 n\pi x/L dx = L/2$, returns (2.3.46) to the more familiar form,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx. \quad (2.3.47)$$

Both formulas (2.3.45) and (2.3.46) are divided by something. In (2.3.45) it is $\mathbf{u} \cdot \mathbf{u}$, or the length of the vector \mathbf{u} squared. Thus, $\int_0^L \sin^2 n\pi x/L dx$ may be thought of as the length squared of $\sin n\pi x/L$ (although here length means nothing other than the square root of the integral). In this manner the length squared of the function $\sin n\pi x/L$ is $L/2$, which is an explanation of the appearance of the term $2/L$ in (2.3.47).

2.4 Worked Examples with the Heat Equation (Other Boundary Value Problems)

2.4.1 Heat Conduction in a Rod with Insulated Ends

Let us work out in detail the solution (and its interpretation) of the following problem defined for $0 \leq x \leq L$ and $t \geq 0$:

$$\text{PDE: } \boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}} \quad (2.4.1)$$

$$\text{BC1: } \boxed{\frac{\partial u}{\partial x}(0, t) = 0} \quad (2.4.2)$$

$$\text{BC2: } \boxed{\frac{\partial u}{\partial x}(L, t) = 0} \quad (2.4.3)$$

$$\text{IC: } \boxed{u(x, 0) = f(x)}. \quad (2.4.4)$$

As a review, this is a heat conduction problem in a one-dimensional rod with constant thermal properties and no sources. This problem is quite similar to the

problem treated in Sec. 2.3, the only difference being the boundary conditions. Here the ends are insulated, whereas in Sec. 2.3 the ends were fixed at 0° . Both the partial differential equation and the boundary conditions are linear and homogeneous. Consequently, we apply the method of separation of variables. We may follow the general procedure described in Sec. 2.3.8. The assumption of product solutions,

$$u(x, t) = \phi(x)G(t), \quad (2.4.5)$$

implies from the PDE as before that

$$\frac{dG}{dt} = -\lambda k G \quad (2.4.6)$$

$$\frac{d^2 \phi}{dx^2} = -\lambda \phi, \quad (2.4.7)$$

where λ is the separation constant. Again,

$$G(t) = ce^{-\lambda kt}. \quad (2.4.8)$$

The insulated boundary conditions, (2.4.2, 2.4.3) imply that the separated solutions must satisfy $d\phi/dx(0) = 0$ and $d\phi/dx(L) = 0$. The separation constant λ is then determined by finding those λ for which nontrivial solutions exist for the following boundary value problem:

$$\boxed{\frac{d^2 \phi}{dx^2} = -\lambda \phi} \quad (2.4.9)$$

$$\boxed{\frac{d\phi}{dx}(0) = 0} \quad (2.4.10)$$

$$\boxed{\frac{d\phi}{dx}(L) = 0}. \quad (2.4.11)$$

Although the ordinary differential equation for the boundary value problem is the same one as previously analyzed, the boundary conditions are different. We must repeat some of the analysis. Once again three cases should be discussed: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$ (since we will assume the eigenvalues are real).

For $\lambda > 0$, the general solution of (2.4.9) is again

$$\phi = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x. \quad (2.4.12)$$

We need to calculate $d\phi/dx$ to satisfy the boundary conditions:

$$\frac{d\phi}{dx} = \sqrt{\lambda} (-c_1 \sin \sqrt{\lambda} x + c_2 \cos \sqrt{\lambda} x). \quad (2.4.13)$$

The boundary condition $d\phi/dx(0) = 0$ implies that $0 = c_2\sqrt{\lambda}$, and hence $c_2 = 0$, since $\lambda > 0$. Thus, $\phi = c_1 \cos \sqrt{\lambda}x$ and $d\phi/dx = -c_1\sqrt{\lambda} \sin \sqrt{\lambda}x$. The eigenvalues λ and their corresponding eigenfunctions are determined from the remaining boundary condition, $d\phi/dx(L) = 0$:

$$0 = -c_1\sqrt{\lambda} \sin \sqrt{\lambda}L.$$

As before, for nontrivial solutions, $c_1 \neq 0$, and hence $\sin \sqrt{\lambda}L = 0$. The eigenvalues for $\lambda > 0$ are the same as the previous problem, $\sqrt{\lambda}L = n\pi$ or

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \dots, \quad (2.4.14)$$

but the corresponding eigenfunctions are cosines (not sines),

$$\phi(x) = c_1 \cos \frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots \quad (2.4.15)$$

The resulting product solutions of the PDE are

$$u(x, t) = A \cos \frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}, \quad n = 1, 2, 3, \dots, \quad (2.4.16)$$

where A is an arbitrary multiplicative constant.

Before applying the principle of superposition, we must see if there are any other eigenvalues. If $\lambda = 0$, then

$$\phi = c_1 + c_2 x, \quad (2.4.17)$$

where c_1 and c_2 are arbitrary constants. The derivative of ϕ is

$$\frac{d\phi}{dx} = c_2.$$

Both boundary conditions, $d\phi/dx(0) = 0$ and $d\phi/dx(L) = 0$, give the same condition, $c_2 = 0$. Thus, there are nontrivial solutions of the boundary value problem for $\lambda = 0$ namely $\phi(x)$ equaling any constant.

$$\phi(x) = c_1. \quad (2.4.18)$$

The time-dependent part is also a constant, since $e^{-\lambda kt}$ for $\lambda = 0$ equals 1. Thus, another product solution of both the linear homogeneous PDE and BCs is $u(x, t) = A$, where A is any constant.

We do not expect there to be any eigenvalues for $\lambda < 0$, since in this case the time-dependent part grows exponentially. In addition, it seems unlikely that we would find a nontrivial linear combination of exponentials which would have a zero slope at both $x = 0$ and $x = L$. In Exercise 2.4.4 you are asked to show that there are no eigenvalues for $\lambda < 0$.

In order to satisfy the initial condition, we use the principle of superposition. We should take a linear combination of *all* product solutions of the PDE (not just those corresponding to $\lambda > 0$). Thus,

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}. \quad (2.4.19)$$

It is interesting to note that this is equivalent to

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}, \quad (2.4.20)$$

since $\cos 0 = 1$ and $e^0 = 1$. In fact, (2.4.20) is often easier to use in practice. We prefer the form (2.4.19) *in the beginning stages of the learning process*, since it more clearly shows that the solution consists of terms arising from the analysis of two somewhat distinct cases, $\lambda = 0$ and $\lambda > 0$.

The initial condition $u(x, 0) = f(x)$ is satisfied if

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}, \quad (2.4.21)$$

for $0 \leq x \leq L$. The validity of (2.4.21) will also follow from the theory of Fourier series. Let us note that in the previous problem $f(x)$ was represented by a series of sines. Here $f(x)$ consists of a series of cosines and the constant term. The two cases are different due to the different boundary conditions. To complete the solution we need to determine the arbitrary coefficients A_0 and A_n ($n \geq 1$). Fortunately, from integral tables it is known that $\cos n\pi x/L$ satisfies the following **orthogonality relation**:

$$\int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \neq 0 \\ L & n = m = 0 \end{cases} \quad (2.4.22)$$

for n and m nonnegative integers. Note that $n = 0$ or $m = 0$ corresponds to a constant 1 contained in the integrand. The constant $L/2$ is another application of the statement that the average of the square of a sine or cosine function is $\frac{1}{2}$. The constant L in (2.4.22) is quite simple since for $n = m = 0$, (2.4.22) becomes $\int_0^L dx = L$. Equation (2.4.22) states that the **cosine functions** (including the constant function) **form an orthogonal set of functions**. We can use that idea, in the same way as before, to determine the coefficients. Multiplying (2.4.21) by $\cos m\pi x/L$ and integrating from 0 to L yields

$$\int_0^L f(x) \cos \frac{m\pi x}{L} dx = \sum_{n=0}^{\infty} A_n \int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx.$$

This holds for all m , $m = 0, 1, 2, \dots$. The case in which $m = 0$ corresponds just to integrating (2.4.21) directly. Using the orthogonality results, it follows that only the m th term in the infinite sum contributes,

$$\int_0^L f(x) \cos \frac{m\pi x}{L} dx = A_m \int_0^L \cos^2 \frac{m\pi x}{L} dx.$$

The factor $\int_0^L \cos^2 m\pi x/L dx$ has two different cases, $m = 0$ and $m \neq 0$. Solving for A_m yields

$$A_0 = \frac{1}{L} \int_0^L f(x) dx \quad (2.4.23)$$

$$(m \geq 1) \quad A_m = \frac{2}{L} \int_0^L f(x) \cos \frac{m\pi x}{L} dx. \quad (2.4.24)$$

The two different formulas are a somewhat annoying feature of this series of cosines. It is simply caused by the factors $L/2$ and L in (2.4.22).

There is a significant difference between the solutions of the PDE for $\lambda > 0$ and the solution for $\lambda = 0$. All the solutions for $\lambda > 0$ decay exponentially in time, whereas the solution for $\lambda = 0$ remains constant in time. Thus, as $t \rightarrow \infty$ the complicated infinite series solution (2.4.19) approaches steady state,

$$\lim_{t \rightarrow \infty} u(x, t) = A_0 = \frac{1}{L} \int_0^L f(x) dx.$$

Not only is the steady-state temperature constant, A_0 , but we recognize the constant A_0 as the average of the initial temperature distribution. This agrees with information obtained previously. Recall from Sec. 1.4 that the equilibrium temperature distribution for the problem with insulated boundaries is not unique. Any constant temperature is an equilibrium solution, but using the ideas of conservation of total thermal energy, we know that the constant must be the average of the initial temperature.

2.4.2 Heat Conduction in a Thin Circular Ring

We have investigated a heat flow problem whose eigenfunctions are sines and one whose eigenfunctions are cosines. In this subsection we illustrate a heat flow problem whose eigenfunctions are both sines and cosines.

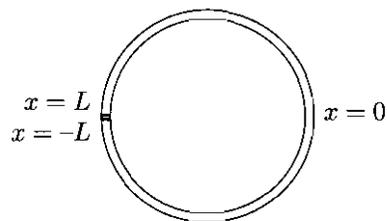


Figure 2.4.1: Thin circular ring.

Let us formulate the appropriate initial boundary value problem if a thin wire (with lateral sides insulated) is bent into the shape of a circle, as illustrated in Fig. 2.4.1. For reasons that will not be apparent for a while, we let the wire have length

$2L$ (rather than L as for the two previous heat conduction problems). Since the circumference of a circle is $2\pi r$, the radius is $r = 2L/2\pi = L/\pi$. If the wire is thin enough, it is reasonable to assume that the temperature in the wire is constant along cross sections of the bent wire. In this situation the wire should satisfy a one-dimensional heat equation, where the distance is actually the arc length x along the wire:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}. \quad (2.4.25)$$

We have assumed that the wire has constant thermal properties and no sources. It is convenient in this problem to measure the arc length x , such that x ranges from $-L$ to $+L$ (instead of the more usual 0 to $2L$).

Let us assume that the wire is very tightly connected to itself at the ends ($x = -L$ to $x = +L$). The conditions of perfect thermal contact should hold there (see Exercise 1.3.2). The temperature $u(x, t)$ is continuous there,

$$u(-L, t) = u(L, t). \quad (2.4.26)$$

Also, since the heat flux must be continuous there (and the thermal conductivity is constant everywhere), the derivative of the temperature is also continuous:

$$\frac{\partial u}{\partial x}(-L, t) = \frac{\partial u}{\partial x}(L, t). \quad (2.4.27)$$

The two boundary conditions for the partial differential equation are (2.4.26) and (2.4.27). The initial condition is that the initial temperature is a given function of the position along the wire,

$$u(x, 0) = f(x). \quad (2.4.28)$$

The mathematical problem consists of the linear homogeneous PDE (2.4.25) subject to linear homogeneous BCs (2.4.26, 2.4.27). As such, we will proceed in the usual way to apply the method of separation of variables. Product solutions $u(x, t) = \phi(x)G(t)$ for the heat equation have been obtained previously, where $G(t) = ce^{-\lambda kt}$. The corresponding boundary value problem is

$$\frac{d^2 \phi}{dx^2} = -\lambda \phi \quad (2.4.29)$$

$$\phi(-L) = \phi(L) \quad (2.4.30)$$

$$\boxed{\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)} \quad (2.4.31)$$

The boundary conditions (2.4.30) and (2.4.31) each involve both boundaries (sometimes called the **mixed** type). The specific boundary conditions (2.4.30) and (2.4.31) are referred to as **periodic boundary conditions** since although the problem can be thought of physically as being defined only for $-L < x < L$, it is often thought of as being defined periodically for all x ; the temperature will be periodic ($x = x_0$ is the same physical point as $x = x_0 + 2L$, and hence must have the same temperature).

If $\lambda > 0$, the general solution of (2.4.29) is again

$$\phi = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x.$$

The boundary condition $\phi(-L) = \phi(L)$ implies that

$$c_1 \cos \sqrt{\lambda}(-L) + c_2 \sin \sqrt{\lambda}(-L) = c_1 \cos \sqrt{\lambda}L + c_2 \sin \sqrt{\lambda}L.$$

Since cosine is an even function, $\cos \sqrt{\lambda}(-L) = \cos \sqrt{\lambda}L$, and since sine is an odd function, $\sin \sqrt{\lambda}(-L) = -\sin \sqrt{\lambda}L$, it follows that $\phi(-L) = \phi(L)$ is satisfied only if

$$c_2 \sin \sqrt{\lambda}L = 0. \quad (2.4.32)$$

Before solving (2.4.32), we analyze the second boundary condition, which involves the derivative,

$$\frac{d\phi}{dx} = \sqrt{\lambda} (-c_1 \sin \sqrt{\lambda}x + c_2 \cos \sqrt{\lambda}x).$$

Thus, $d\phi/dx(-L) = d\phi/dx(L)$ is satisfied only if

$$c_1 \sqrt{\lambda} \sin \sqrt{\lambda}L = 0, \quad (2.4.33)$$

where the evenness of cosines and the oddness of sines have again been used. Conditions (2.4.32) and (2.4.33) are easily solved. If $\sin \sqrt{\lambda}L \neq 0$, then $c_1 = 0$ and $c_2 = 0$, which is just the trivial solution. Thus, for nontrivial solutions,

$$\sin \sqrt{\lambda}L = 0,$$

which determines the eigenvalues λ . We find (as before) that $\sqrt{\lambda}L = n\pi$ or equivalently that

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \dots \quad (2.4.34)$$

We chose the wire to have length $2L$ so that the eigenvalues have the same formula as before (this will mean less to remember, as all our problems have a similar answer). However, in this problem (unlike the others) there are no additional constraints that c_1 and c_2 must satisfy. Both are arbitrary. We say that both $\sin n\pi x/L$ and $\cos n\pi x/L$ are eigenfunctions corresponding to the eigenvalue $\lambda = (n\pi/L)^2$,

$$\phi(x) = \cos \frac{n\pi x}{L}, \sin \frac{n\pi x}{L}, n = 1, 2, 3, \dots \quad (2.4.35)$$

In fact, any linear combination of $\cos n\pi x/L$ and $\sin n\pi x/L$ is an eigenfunction,

$$\phi(x) = c_1 \cos \frac{n\pi x}{L} + c_2 \sin \frac{n\pi x}{L}, \quad (2.4.36)$$

but this is always to be understood when the statement is made that both are eigenfunctions. There are thus two infinite families of product solutions of the partial differential equation, $n = 1, 2, 3, \dots$,

$$u(x, t) = \cos \frac{n\pi x}{L} e^{-(n\pi/L)^2 kt} \quad \text{and} \quad u(x, t) = \sin \frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}. \quad (2.4.37)$$

All of these correspond to $\lambda > 0$.

If $\lambda = 0$, the general solution of (2.4.29) is

$$\phi = c_1 + c_2 x.$$

The boundary condition $\phi(-L) = \phi(L)$ implies that

$$c_1 - c_2 L = c_1 + c_2 L.$$

Thus, $c_2 = 0$, $\phi(x) = c_1$ and $d\phi/dx = 0$. The remaining boundary condition, (2.4.30), is automatically satisfied. We see that

$$\phi(x) = c_1,$$

any constant, is an eigenfunction, corresponding to the eigenvalue zero. Sometimes we say that $\phi(x) = 1$ is the eigenfunction, since it is known that any multiple of an eigenfunction is always an eigenfunction. Product solutions $u(x, t)$ are also constants in this case. Note that there is only one independent eigenfunction corresponding to $\lambda = 0$, while for each positive eigenvalue in this problem, $\lambda = (n\pi/L)^2$, there are two independent eigenfunctions, $\sin n\pi x/L$ and $\cos n\pi x/L$. Not surprisingly, it can be shown that there are no eigenvalues in which $\lambda < 0$.

The principle of superposition must be used before applying the initial condition. The most general solution obtainable by the method of separation of variables consists of an arbitrary linear combination of all product solutions:

$$\boxed{u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} e^{-(n\pi/L)^2 kt} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}. \quad (2.4.38)}$$

The constant a_0 is the product solution corresponding to $\lambda = 0$, whereas two families of arbitrary coefficients, a_n and b_n , are needed for $\lambda > 0$. The initial condition $u(x, 0) = f(x)$ is satisfied if

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \quad (2.4.39)}$$

Here the function $f(x)$ is a linear combination of both sines and cosines (and a constant), unlike the previous problems, where either sines or cosines (including the constant term) were used. Another crucial difference is that (2.4.39) should be valid for the entire ring, which means that $-L \leq x \leq L$, whereas the series of just sines or cosines was valid for $0 \leq x \leq L$. The theory of Fourier series will show that (2.4.39) is valid, and, more important, that the previous series of just sines or cosines are but special cases of the series in (2.4.39).

For now, we wish just to determine the coefficients a_0 , a_n , b_n from (2.4.39). Again the eigenfunctions form an orthogonal set since integral tables verify the following orthogonality conditions:

$$\int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \\ 2L & n = m = 0 \end{cases} \quad (2.4.40)$$

$$\int_{-L}^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \end{cases} \quad (2.4.41)$$

$$\int_{-L}^L \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = 0, \quad (2.4.42)$$

where n and m are arbitrary (nonnegative) integers. The constant eigenfunction corresponds to $n = 0$ or $m = 0$. Integrals of the square of sines or cosines ($n = m$) are evaluated again by the "half the length of the interval" rule. The last of these formulas, (2.4.42), is particularly simple to derive, since sine is an odd function and cosine is an even function.⁴ Note that, for example, $\cos n\pi x/L$ is orthogonal to every other eigenfunction [sines from (2.4.42), cosines and the constant eigenfunction from (2.4.40)].

The coefficients are derived in the same manner as before. A few steps are saved by noting (2.4.39) is equivalent to

$$f(x) = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

If we multiply this by both $\cos m\pi x/L$ and $\sin m\pi x/L$ and then integrate from

⁴The product of an odd and an even function is odd. By antisymmetry the integral of an odd function over a symmetric interval is zero.

$x = -L$ to $x = +L$, we obtain

$$\int_{-L}^L f(x) \begin{Bmatrix} \cos \frac{m\pi x}{L} \\ \sin \frac{m\pi x}{L} \end{Bmatrix} dx = \sum_{n=0}^{\infty} a_n \int_{-L}^L \cos \frac{n\pi x}{L} \begin{Bmatrix} \cos \frac{m\pi x}{L} \\ \sin \frac{m\pi x}{L} \end{Bmatrix} dx + \sum_{n=1}^{\infty} b_n \int_{-L}^L \sin \frac{n\pi x}{L} \begin{Bmatrix} \cos \frac{m\pi x}{L} \\ \sin \frac{m\pi x}{L} \end{Bmatrix} dx.$$

If we utilize (2.4.40)-(2.4.42), we find that

$$\int_{-L}^L f(x) \cos \frac{m\pi x}{L} dx = a_m \int_{-L}^L \cos^2 \frac{m\pi x}{L} dx$$

$$\int_{-L}^L f(x) \sin \frac{m\pi x}{L} dx = b_m \int_{-L}^L \sin^2 \frac{m\pi x}{L} dx.$$

Solving for the coefficients in a manner that we are now familiar with yields

$$\begin{cases} a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx \\ a_m = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{m\pi x}{L} dx \\ b_m = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{m\pi x}{L} dx. \end{cases} \quad (m \geq 1) \quad (2.4.43)$$

The solution to the problem is (2.4.38), where the coefficients are given by (2.4.43).

2.4.3 Summary of Boundary Value Problems

In many problems, including the ones we have just discussed, the specific simple constant-coefficient differential equation,

$$\frac{d^2 \phi}{dx^2} = -\lambda \phi,$$

forms the fundamental part of the boundary value problem. Above we have collected in one place the relevant formulas for the eigenvalues and eigenfunctions for the typical boundary conditions already discussed. You will find it helpful to understand these results because of their enormous applicability throughout this text.

BOUNDARY VALUE PROBLEMS FOR $\frac{d^2\phi}{dx^2} = -\lambda\phi$

Boundary conditions	$\phi(0) = 0$ $\phi(L) = 0$	$\frac{d\phi}{dx}(0) = 0$ $\frac{d\phi}{dx}(L) = 0$	$\phi(-L) = \phi(L)$ $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$
Eigenvalues λ_n	$\left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, 3, \dots$	$\left(\frac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \dots$	$\left(\frac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \dots$
Eigenfunctions	$\sin \frac{n\pi x}{L}$	$\cos \frac{n\pi x}{L}$	$\sin \frac{n\pi x}{L}$ and $\cos \frac{n\pi x}{L}$
Series	$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$
Coefficients	$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$	$A_0 = \frac{1}{L} \int_0^L f(x) dx$ $A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$	$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$ $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$ $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$

It is important to note that, in these cases, whenever $\lambda = 0$ is an eigenvalue, a constant is the eigenfunction (corresponding to $n = 0$ in $\cos n\pi x/L$).

EXERCISES 2.4

*2.4.1. Solve the heat equation $\partial u/\partial t = k\partial^2 u/\partial x^2$, $0 < x < L$, $t > 0$, subject to

$$\frac{\partial u}{\partial x}(0, t) = 0 \quad t > 0 \quad \frac{\partial u}{\partial x}(L, t) = 0 \quad t > 0.$$

$$(a) \quad u(x, 0) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases} \quad (b) \quad u(x, 0) = 6 + 4 \cos \frac{3\pi x}{L}$$

$$(c) \quad u(x, 0) = -2 \sin \frac{\pi x}{L} \quad (d) \quad u(x, 0) = -3 \cos \frac{8\pi x}{L}$$

*2.4.2. Solve

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \frac{\partial u}{\partial x}(0, t) = 0$$

$$u(L, t) = 0$$

$$u(x, 0) = f(x).$$

For this problem you may assume that no solutions of the heat equation exponentially grow in time. You may also guess appropriate orthogonality conditions for the eigenfunctions.

*2.4.3. Solve the eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

$$\text{subject to } \phi(0) = \phi(2\pi) \quad \text{and} \quad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(2\pi).$$

2.4.4. Explicitly show there are no negative eigenvalues for

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad \text{subject to} \quad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(L) = 0.$$

2.4.5. This problem presents an alternative derivation of the heat equation for a thin wire. The equation for a circular wire of finite thickness is the two-dimensional heat equation (in polar coordinates). Show that this reduces to (2.4.25) if the temperature does not depend on r and if the wire is very thin.

2.4.6. Determine the equilibrium temperature distribution for the thin circular ring of Section 2.4.2:

- Directly from the equilibrium problem (see Sec. 1.4).
- By computing the limit as $t \rightarrow \infty$ of the time-dependent problem.

2.4.7. Solve Laplace's equation inside a circle of radius a ,

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0,$$

subject to the boundary condition $u(a, \theta) = f(\theta)$. (*Hint:* If necessary, see Sec. 2.5.2.)

2.5 Laplace's Equation: Solutions and Qualitative Properties

2.5.1 Laplace's Equation inside a Rectangle

In order to obtain more practice, we will consider a different kind of problem which can be analyzed by the method of separation of variables. We consider steady-state heat conduction in a two-dimensional region. To be specific, consider the equilibrium temperature inside a rectangle ($0 \leq x \leq L$, $0 \leq y \leq H$) when the temperature is a prescribed function of position (independent of time) on the boundary.

The equilibrium temperature $u(x, y)$ satisfies Laplace's equation with the following boundary conditions:

$$\text{PDE: } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (2.5.1)$$

$$\text{BC1: } u(0, y) = g_1(y) \quad (2.5.2)$$

$$\text{BC2: } u(L, y) = g_2(y) \quad (2.5.3)$$

$$\text{BC3: } u(x, 0) = f_1(x) \quad (2.5.4)$$

$$\text{BC4: } u(x, H) = f_2(x), \quad (2.5.5)$$

where $f_1(x)$, $f_2(x)$, $g_1(y)$, and $g_2(y)$ are given functions of x and y , respectively. Here the partial differential equation is linear and homogeneous, but the boundary conditions, although linear, are not homogeneous. We will not be able to apply the method of separation of variables to this problem in its present form, because when we separate variables the boundary value problem (determining the separation constant) must have homogeneous boundary conditions. In this example all the boundary conditions are nonhomogeneous. We can get around this difficulty by noting that the original problem is nonhomogeneous due to the four nonhomogeneous boundary conditions. The idea behind the principle of superposition can be used sometimes for nonhomogeneous problems (see Exercise 2.2.4). We break our problem up into four problems each having one nonhomogeneous condition. We let

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y), \quad (2.5.6)$$

where each $u_i(x, y)$ satisfies Laplace's equation with one nonhomogeneous boundary condition and the related three homogeneous boundary conditions, as diagrammed in Fig. 2.5.1. Instead of directly solving for u , we will indicate how to solve for u_1 , u_2 , u_3 , and u_4 . Why does the sum satisfy our problem? We check to see that the PDE and the four nonhomogeneous BCs will be satisfied. Since u_1 , u_2 , u_3 , and u_4 satisfy Laplace's equation, which is linear and homogeneous, $u \equiv u_1 + u_2 + u_3 + u_4$ also satisfies the same linear and homogeneous PDE by the principle of superposition. At $x = 0$, $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, and $u_4 = g_1(y)$. Therefore, at $x = 0$, $u = u_1 + u_2 + u_3 + u_4 = g_1(y)$, the desired nonhomogeneous condition. In a similar manner we can check that all four nonhomogeneous conditions have been satisfied.

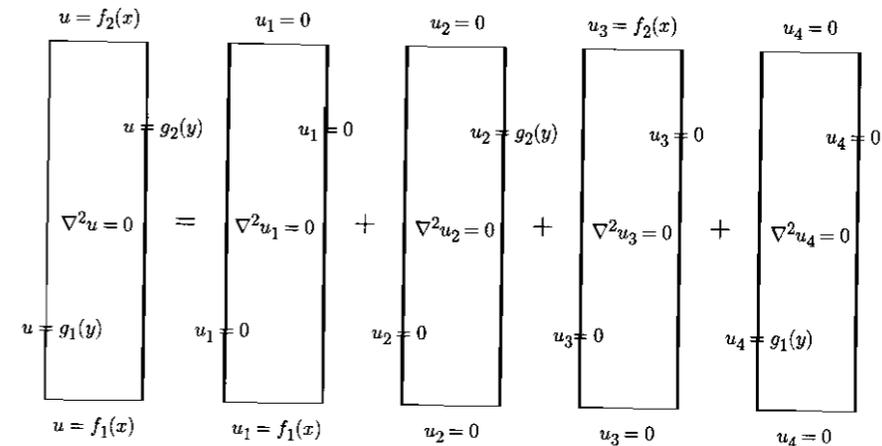


Figure 2.5.1: Laplace's equation inside a rectangle.

The method to solve for any of the $u_i(x, y)$ is the same; only certain details differ. We will only solve for $u_4(x, y)$, and leave the rest for the exercises:

$$\text{PDE: } \frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0 \quad (2.5.7)$$

$$\text{BC1: } u_4(0, y) = g_1(y) \quad (2.5.8)$$

$$\text{BC2: } u_4(L, y) = 0 \quad (2.5.9)$$

$$\text{BC3: } u_4(x, 0) = 0 \quad (2.5.10)$$

$$\text{BC4: } u_4(x, H) = 0, \quad (2.5.11)$$

We propose to solve this problem by the method of separation of variables. We begin by ignoring the nonhomogeneous condition $u_4(0, y) = g_1(y)$. Eventually, we will add together product solutions to synthesize $g_1(y)$. We look for product solutions

$$u_4(x, y) = h(x)\phi(y). \quad (2.5.12)$$

From the three homogeneous boundary conditions, we see that

$$h(L) = 0 \quad (2.5.13)$$

$$\phi(0) = 0 \quad (2.5.14)$$

$$\phi(H) = 0. \quad (2.5.15)$$

Thus, the y -dependent solution $\phi(y)$ has two homogeneous boundary conditions, whereas the x -dependent solution $h(x)$ has only one. If (2.5.12) is substituted into Laplace's equation, we obtain

$$\phi(y) \frac{d^2 h}{dx^2} + h(x) \frac{d^2 \phi}{dy^2} = 0.$$

The variables can be separated by dividing by $h(x)\phi(y)$, so that

$$\frac{1}{h} \frac{d^2 h}{dx^2} = -\frac{1}{\phi} \frac{d^2 \phi}{dy^2}. \quad (2.5.16)$$

The left-hand side is only a function of x , while the right-hand side is only a function of y . Both must equal a separation constant. Do we want to use $-\lambda$ or λ ? One will be more convenient. If the separation constant is negative (as it was before), (2.5.16) implies that $h(x)$ oscillates and $\phi(y)$ is composed of exponentials. This seems doubtful, since the homogeneous boundary conditions (2.5.13)-(2.5.15) show that the y -dependent solution satisfies two homogeneous conditions; $\phi(y)$ must be zero at $y = 0$ and at $y = H$. Exponentials in y are not expected to work. On the other hand, if the separation constant is positive, (2.5.16) implies that $h(x)$ is composed of exponentials and $\phi(y)$ oscillates. This seems more reasonable, and we thus introduce the separation constant λ (but we do *not* assume $\lambda \geq 0$):

$$\frac{1}{h} \frac{d^2 h}{dx^2} = -\frac{1}{\phi} \frac{d^2 \phi}{dy^2} = \lambda. \quad (2.5.17)$$

This results in two ordinary differential equations:

$$\begin{aligned} \frac{d^2 h}{dx^2} &= \lambda h \\ \frac{d^2 \phi}{dy^2} &= -\lambda \phi. \end{aligned}$$

The x -dependent problem is *not* a boundary value problem, since it does not have two homogeneous boundary conditions.

$$\boxed{\frac{d^2 h}{dx^2} = \lambda h} \quad (2.5.18)$$

$$\boxed{h(L) = 0.} \quad (2.5.19)$$

However, the y -dependent problem is a boundary value problem and will be used to determine the eigenvalues λ (separation constants):

$$\boxed{\frac{d^2 \phi}{dy^2} = -\lambda \phi} \quad (2.5.20)$$

$$\boxed{\phi(0) = 0} \quad (2.5.21)$$

$$\boxed{\phi(H) = 0.} \quad (2.5.22)$$

This boundary value problem is one that has arisen before, but here the length of the interval is H . All the eigenvalues are positive, $\lambda > 0$. The eigenfunctions are clearly sines, since $\phi(0) = 0$. Furthermore, the condition $\phi(H) = 0$ implies that

$$\boxed{\begin{aligned} \lambda &= \left(\frac{n\pi}{H}\right)^2 \\ \phi(y) &= \sin \frac{n\pi y}{H} \end{aligned}} \quad n = 1, 2, 3, \dots \quad (2.5.23)$$

To obtain product solutions we now must solve (2.5.18) with (2.5.19). Since $\lambda = (n\pi/H)^2$,

$$\frac{d^2 h}{dx^2} = \left(\frac{n\pi}{H}\right)^2 h. \quad (2.5.24)$$

The general solution is a linear combination of exponentials or a linear combination of hyperbolic functions. Either can be used, but neither is particularly suited for solving the homogeneous boundary condition $h(L) = 0$. We can obtain our solution more expeditiously, if we note that both $\cosh n\pi(x-L)/H$ and $\sinh n\pi(x-L)/H$ are linearly independent solutions of (2.5.24). The general solution can be written as a linear combination of these two:

$$h(x) = a_1 \cosh \frac{n\pi}{H}(x-L) + a_2 \sinh \frac{n\pi}{H}(x-L), \quad (2.5.25)$$

although it should now be clear that $h(L) = 0$ implies that $a_1 = 0$ (since $\cosh 0 = 1$ and $\sinh 0 = 0$). As we could have guessed originally,

$$h(x) = a_2 \sinh \frac{n\pi}{H}(x-L). \quad (2.5.26)$$

The reason (2.5.25) is the solution (besides the fact that it solves the DE) is that it is a simple translation of the more familiar solution, $\cosh n\pi x/L$ and $\sinh n\pi x/L$. We are allowed to translate solutions of differential equations only if the differential equation does not change (said to be *invariant*) upon translation. Since (2.5.24)

has constant coefficients, thinking of the origin being at $x = L$ (namely $x' = x - L$) does not affect the differential equation. since $d^2h/dx'^2 = (n\pi/H)^2h$ according to the chain rule. For example $\cosh n\pi x'/H = \cosh n\pi(x - L)/H$ is a solution.

Product solutions are

$$u_4(x, y) = A \sin \frac{n\pi y}{H} \sinh \frac{n\pi}{H}(x - L). \quad (2.5.27)$$

You might now check that Laplace's equation is satisfied as well as the three required homogeneous conditions. It is interesting to note that one part (the y) oscillates and the other (the x) does not. This is a general property of Laplace's equation, not restricted to this geometry (rectangle) or to these boundary conditions.

We want to use these product solutions to satisfy the remaining condition, the nonhomogeneous boundary condition $u_4(0, y) = g_1(y)$. Product solutions do *not* satisfy nonhomogeneous conditions. Instead, we again use the principle of superposition. If (2.5.27) is a solution, so is

$$u_4(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{H} \sinh \frac{n\pi}{H}(x - L). \quad (2.5.28)$$

Evaluating at $x = 0$ will determine the coefficients A_n from the nonhomogeneous boundary condition:

$$g_1(y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{H} \sinh \frac{n\pi}{H}(-L).$$

This is the same kind of series of sine functions we have already briefly discussed, if we associate $A_n \sinh n\pi(-L)/H$ as its coefficients. Thus (by the orthogonality of $\sin n\pi y/H$ for y between 0 and H),

$$A_n \sinh \frac{n\pi}{H}(-L) = \frac{2}{H} \int_0^H g_1(y) \sin \frac{n\pi y}{H} dy.$$

Since $\sinh n\pi(-L)/H$ is never zero, we can divide by it and obtain finally a formula for the coefficients:

$$A_n = \frac{2}{H \sinh n\pi(-L)/H} \int_0^H g_1(y) \sin \frac{n\pi y}{H} dy. \quad (2.5.29)$$

Equation (2.5.28) with coefficients determined by (2.5.29) is only the solution for $u_4(x, y)$. The original $u(x, y)$ is obtained by adding together four such solutions.

2.5.2 Laplace's Equation for a Circular Disk

Suppose that we have a thin circular disk of radius a (with constant thermal properties and no sources) with the temperature prescribed on the boundary as illustrated in Fig. 2.5.2. If the temperature on the boundary is independent of time, then it is reasonable to determine the equilibrium temperature distribution. The temperature satisfies Laplace's equation, $\nabla^2 u = 0$. The geometry of this problem suggests that we use polar coordinates, so that $u = u(r, \theta)$. In particular, on the circle $r = a$ the temperature distribution is a prescribed function of θ , $u(a, \theta) = f(\theta)$. The problem we want to solve is

$$\text{PDE: } \nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad (2.5.30)$$

$$\text{BC: } u(a, \theta) = f(\theta). \quad (2.5.31)$$

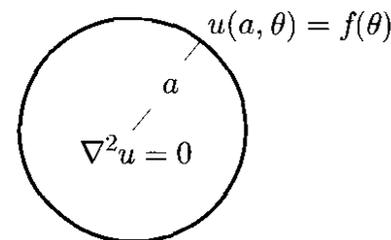


Figure 2.5.2: Laplace's equation inside a circular disk.

At first glance, it would appear that we cannot use separation of variables because there are no homogeneous subsidiary conditions. However, the introduction of polar coordinates requires some discussion that will illuminate the use of the method of separation of variables. If we solve Laplace's equation on a rectangle (see Sec. 2.5.1), $0 \leq x \leq L$, $0 \leq y \leq H$, then conditions are necessary at the endpoints of definition of the variables, $x = 0, L$ and $y = 0, H$. Fortunately, these coincide with the physical boundaries. However, for polar coordinates, $0 \leq r \leq a$ and $-\pi \leq \theta \leq \pi$ (where there is some freedom in our definition of the angle θ). Mathematically, we need conditions at the endpoints of the coordinate system, $r = 0, a$ and $\theta = -\pi, \pi$. Here, only $r = a$ corresponds to a physical boundary. Thus, we need conditions motivated by considerations of the physical problem at $r = 0$ and at $\theta = \pm\pi$. Polar coordinates are singular at $r = 0$; for physical reasons we will prescribe that the temperature is finite or, equivalently, **bounded** there:

$$\text{boundedness at origin } |u(0, \theta)| < \infty. \quad (2.5.32)$$

Conditions are needed at $\theta = \pm\pi$ for mathematical reasons. It is similar to the circular wire situation. $\theta = -\pi$ corresponds to the same points as $\theta = \pi$. Although

there really is not a boundary, we say that the temperature is continuous there and the heat flow in the θ -direction is continuous, which imply:

$$\text{periodicity} \quad \begin{array}{l} u(r, -\pi) = u(r, \pi) \\ \frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi), \end{array} \quad (2.5.33)$$

as though the two regions were in perfect thermal contact there (see Exercise 1.3.2). Equations (2.5.33) are called **periodicity conditions**; they are equivalent to $u(r, \theta) = u(r, \theta + 2\pi)$. We note that subsidiary conditions (2.5.32) and (2.5.33) are all linear and homogeneous (it's easy to check that $u \equiv 0$ satisfies these three conditions). In this form the mathematical problem appears somewhat similar to Laplace's equation inside a rectangle. There are four conditions. Here, fortunately, only one is nonhomogeneous, $u(a, \theta) = f(\theta)$. This problem is thus suited for the method of separation of variables.

We look for special product solutions,

$$u(r, \theta) = \phi(\theta)G(r), \quad (2.5.34)$$

which satisfy the PDE (2.5.30) and the three homogeneous conditions (2.5.32) and (2.5.33). Note that (2.5.34) does *not* satisfy the nonhomogeneous boundary condition (2.5.31). Substituting (2.5.34) into the periodicity conditions shows that

$$\begin{array}{l} \phi(-\pi) = \phi(\pi) \\ \frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi); \end{array} \quad (2.5.35)$$

the θ -dependent part also satisfies the **periodic boundary conditions**. The product form will satisfy Laplace's equation if

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dG}{dr} \right) \phi(\theta) + \frac{1}{r^2} G(r) \frac{d^2\phi}{d\theta^2} = 0.$$

The variables are *not* separated by dividing by $G(r)\phi(\theta)$ since $1/r^2$ remains multiplying the θ -dependent terms. Instead, divide by $(1/r^2)G(r)\phi(\theta)$, in which case,

$$\frac{r}{G} \frac{d}{dr} \left(r \frac{dG}{dr} \right) = -\frac{1}{\phi} \frac{d^2\phi}{d\theta^2} = \lambda. \quad (2.5.36)$$

The separation constant is introduced as λ (rather than $-\lambda$) since there are two homogeneous conditions in θ , (2.5.35), and we therefore expect oscillations in θ . Equation (2.5.36) yields two ordinary differential equations. The boundary value

problem to determine the separation constant is

$$\begin{array}{l} \frac{d^2\phi}{d\theta^2} = -\lambda\phi \\ \phi(-\pi) = \phi(\pi) \\ \frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi). \end{array} \quad (2.5.37)$$

The eigenvalues λ are determined in the usual way. In fact, this is one of the three standard problems, the identical problem as for the circular wire (with $L = \pi$). Thus, the eigenvalues are

$$\lambda = \left(\frac{n\pi}{L} \right)^2 = n^2, \quad (2.5.38)$$

with the corresponding eigenfunctions being both

$$\sin n\theta \quad \text{and} \quad \cos n\theta. \quad (2.5.39)$$

The case $n = 0$ must be included (with only a constant being the eigenfunction).

The r -dependent problem is

$$\frac{r}{G} \frac{d}{dr} \left(r \frac{dG}{dr} \right) = \lambda n^2, \quad (2.5.40)$$

which when written in the more usual form becomes

$$r^2 \frac{d^2G}{dr^2} + r \frac{dG}{dr} - n^2 G = 0. \quad (2.5.41)$$

Here, the condition at $r = 0$ has already been discussed. We have prescribed $|u(0, \theta)| < \infty$. For the product solutions, $u(r, \theta) = \phi(\theta)G(r)$, it follows that the condition at the origin is that $G(r)$ must be bounded there,

$$|G(0)| < \infty. \quad (2.5.42)$$

Equation (2.5.41) is linear and homogeneous but has nonconstant coefficients. There are exceedingly few second-order linear equations with nonconstant coefficients that we can solve easily. Equation (2.5.41) is one such case, an example of an equation known by a number of different names: **equidimensional** or **Cauchy** or **Euler**. The simplest way to solve (2.5.41) is to note that for the linear differential operator in (2.5.41) any power $G = r^p$ reproduces itself.⁵ On substituting $G = r^p$ into (2.5.41) we determine that $[p(p-1) + p - n^2]r^p = 0$. Thus, there are usually two distinct solutions

$$p = \pm n,$$

⁵For constant-coefficient linear differential operators, exponentials reproduce themselves.

except when $n = 0$, in which case there is only one independent solution in the form r^n . For $n \neq 0$, the general solution of (2.5.41) is

$$G = c_1 r^n + c_2 r^{-n}. \quad (2.5.43)$$

For $n = 0$ (and $n = 0$ is important since $\lambda = 0$ is an eigenvalue in this problem), one solution is $r^0 = 1$ or any constant. A second solution for $n = 0$ is most easily obtained from (2.5.40). If $n = 0$, $\frac{d}{dr} \left(r \frac{dG}{dr} \right) = 0$. By integration, $r \frac{dG}{dr}$ is constant, or equivalently dG/dr is proportional to $1/r$. The second independent solution is thus $\ln r$. Thus, for $n = 0$, the general solution of (2.5.41) is

$$G = \bar{c}_1 + \bar{c}_2 \ln r. \quad (2.5.44)$$

Equation (2.5.41) has only one homogeneous condition to be imposed, $|G(0)| < \infty$, so it is not an eigenvalue problem. The boundedness condition would not have imposed any restrictions on the problems we have studied previously. However, here (2.5.43) or (2.5.44) shows that solutions may approach ∞ as $r \rightarrow 0$. Thus, for $|G(0)| < \infty$, $c_2 = 0$ in (2.5.43) and $\bar{c}_2 = 0$ in (2.5.44). The r -dependent solution (which is bounded at $r = 0$) is

$$G(r) = c_1 r^n \quad n \geq 0,$$

where for $n = 0$ this reduces to just an arbitrary constant.

Product solutions by the method of separation of variables, which satisfy the three homogeneous conditions, are

$$r^n \cos n\theta (n \geq 0) \quad \text{and} \quad r^n \sin n\theta (n \geq 1).$$

Note that as in rectangular coordinates for Laplace's equation, oscillations occur in one variable (here θ) and do not occur in the other variable (r). By the principle of superposition, the following solves Laplace's equation inside a circle:

$$\boxed{u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta, \quad \begin{matrix} 0 \leq r < a \\ -\pi < \theta \leq \pi. \end{matrix}} \quad (2.5.45)$$

In order to solve the nonhomogeneous condition, $u(a, \theta) = f(\theta)$,

$$f(\theta) = \sum_{n=0}^{\infty} A_n a^n \cos n\theta + \sum_{n=1}^{\infty} B_n a^n \sin n\theta, \quad -\pi < \theta \leq \pi. \quad (2.5.46)$$

The prescribed temperature is a linear combination of all sines and cosines (including a constant term, $n = 0$). This is exactly the same question that we answered in Sec. 2.4.2 with $L = \pi$ if we let $A_n a^n$ be the coefficient of $\cos n\theta$ and $B_n a^n$ be the

coefficient of $\sin n\theta$. Using the orthogonality formulas it follows that

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta \\ (n \geq 1) \quad A_n a^n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta \\ B_n a^n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta. \end{aligned} \quad (2.5.47)$$

Since $a^n \neq 0$, the coefficients A_n and B_n can be uniquely solved for from (2.5.47).

Equation (2.5.45) with coefficients given by (2.5.47) determines the steady-state temperature distribution inside a circle. The solution is relatively complicated, often requiring the numerical evaluation of two infinite series. For additional interpretations of this solution, see Chapter 8 on Green's functions.

2.5.3 Fluid Flow Past a Circular Cylinder (Lift)

In heat flow, conservation of thermal energy can be used to derive Laplace's equation $\nabla^2 u = 0$ under certain assumptions. In fluid dynamics, conservation of mass and conservation of momentum can be used to also derive Laplace's equation:

$$\nabla^2 \psi = 0,$$

in the following way. In the exercises, it is shown that **conservation of mass** for a fluid along with the assumption of a constant mass density ρ yields

$$\nabla \cdot \mathbf{u} = 0 \quad \text{or in two dimensions} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (2.5.48)$$

where the **velocity** has x and y components $\mathbf{u} = (u, v)$. A **stream function** ψ is often introduced which automatically satisfies (2.5.48):

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \quad (2.5.49)$$

Often streamlines ($\psi = \text{constant}$) are graphed which will be parallel to the fluid flow. It can be shown that in some circumstances the fluid is irrotational ($\nabla \times \mathbf{u} = 0$) so that the stream function satisfies Laplace's equation

$$\boxed{\nabla^2 \psi = 0.} \quad (2.5.50)$$

The simplest example is a constant flow in the x -direction $\mathbf{u} = (U, 0)$, in which case the stream function is $\psi = Uy$, clearly satisfying Laplace's equation.

As a first step in designing airplane wings, scientists have considered the flow around a circular cylinder of radius a . For more details we refer the interested reader to Acheson [1990]. The stream function must satisfy Laplace's equation which as

before in polar coordinate is (2.5.30). We will assume that far from the cylinder the flow is uniform so that as an approximation for large r :

$$\psi \approx Uy = Ur \sin \theta, \quad (2.5.51)$$

since we will use polar coordinates. The boundary condition is that the radial component of the fluid flow must be zero at $r = a$. The fluid flow must be parallel to the boundary, and hence we can assume:

$$\psi(a, \theta) = 0. \quad (2.5.52)$$

By separation of variables, including the $n = 0$ case given by (2.5.44),

$$\psi(r, \theta) = c_1 + c_2 \ln r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \sin n\theta, \quad (2.5.53)$$

where the $\cos n\theta$ terms could be included (but would vanish). By applying the boundary condition at $r = a$, we find

$$\begin{aligned} c_1 + c_2 \ln a &= 0 \\ A_n a^n + B_n a^{-n} &= 0, \end{aligned}$$

so that

$$\psi(r, \theta) = c_1 \ln \frac{r}{a} + \sum_{n=1}^{\infty} A_n \left(r^n - \frac{a^{2n}}{r^n} \right) \sin n\theta. \quad (2.5.54)$$

In order for the fluid velocity to be approximately a constant at infinity with $\psi \approx Uy = Ur \sin \theta$ for large r , $A_n = 0$ for $n \geq 2$ and $A_1 = U$. Thus,

$$\psi(r, \theta) = c_1 \ln \frac{r}{a} + U \left(r - \frac{a^2}{r} \right) \sin \theta. \quad (2.5.55)$$

It can be shown in general that the fluid velocity in polar coordinates can be obtained from the stream function: $u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$, $u_\theta = -\frac{\partial \psi}{\partial r}$. Thus, the θ -component of the fluid velocity is $u_\theta = -\frac{c_1}{r} - U \left(1 + \frac{a^2}{r^2} \right) \sin \theta$. The **circulation** is defined to be $\int_0^{2\pi} u_\theta r d\theta = -2\pi c_1$. For a given velocity at infinity, different flows depending on the circulation around a cylinder are illustrated in Figure 2.5.3.

The **pressure** p of the fluid exerts a force in the direction opposite to the outward normal to the cylinder $\left(\frac{x}{a}, \frac{y}{a} \right) = (\cos \theta, \sin \theta)$. The **drag** (x -direction) and **lift** (y -direction) forces (per unit length in the z -direction) exerted by the fluid on the cylinder are

$$\mathbf{F} = - \int_0^{2\pi} p (\cos \theta, \sin \theta) a d\theta. \quad (2.5.56)$$

For steady flows such as this one, the pressure is determined from **Bernoulli's condition**

$$p + \frac{1}{2} \rho |\mathbf{u}|^2 = \text{constant}. \quad (2.5.57)$$

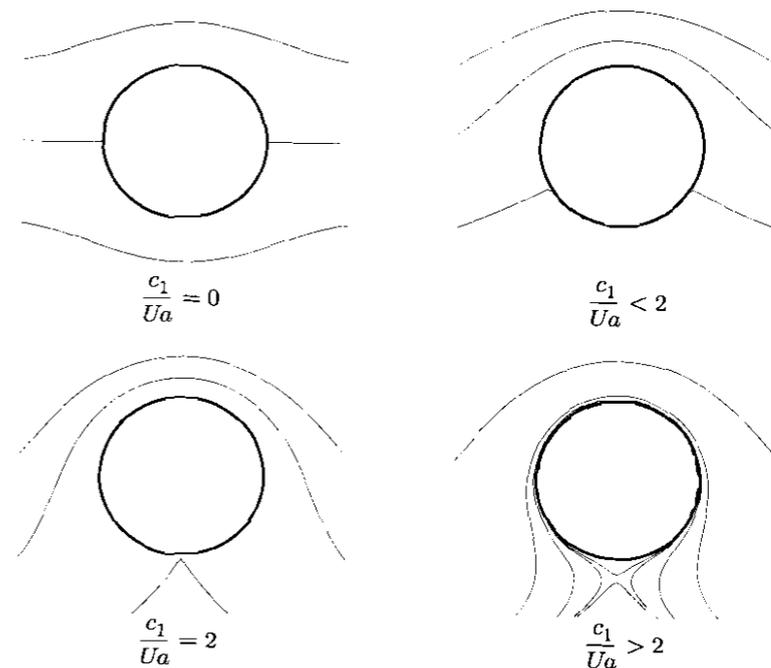


Figure 2.5.3: Flow past cylinder and lift $= 2\pi \rho c_1 U$.

Thus, the pressure is lower where the velocity is higher. If the circulation is clockwise around the cylinder (a negative circulation), then intuitively (which can be verified) the velocity will be higher above the cylinder than below and the pressure will be lower on the top of the cylinder and hence lift (a positive force in the y -direction) will be generated. At the cylinder $u_r = 0$, so that there $|\mathbf{u}|^2 = u_\theta^2$. It can be shown that the x -component of the force, the drag, is zero, but the y -component the lift is given by (since the integral involving the constant vanishes):

$$F_y = \frac{1}{2} \rho \int_0^{2\pi} \left[-\frac{c_1}{r} - U \left(1 + \frac{a^2}{r^2} \right) \sin \theta \right]^2 \sin \theta a d\theta. \quad (2.5.58)$$

$$F_y = \rho \frac{c_1}{a} U 2 \int_0^{2\pi} \sin^2 \theta a d\theta = \rho 2\pi c_1 U, \quad (2.5.59)$$

which has been simplified since $\int_0^{2\pi} \sin \theta d\theta = \int_0^{2\pi} \sin^3 \theta d\theta = 0$ due to the oddness of the sine function. The lift vanishes if the circulation is zero. A negative circulation (positive c_1) results in a lift force on the cylinder by the fluid.

In the real world the drag is more complicated. Boundary layers exist due to the viscous nature of the fluid. The pressure is continuous across the boundary layer so that the above analysis is still often valid. However, things get much more complicated when the boundary layer separates from the cylinder in which case a more substantial drag force occurs (which has been ignored in this elementary

treatment). A plane will fly if the lift is greater than the weight of the plane. However, to fly fast a powerful engine is necessary to apply a force in the x -direction to overcome the drag.

2.5.4 Qualitative Properties of Laplace's Equation

Sometimes the method of separation of variables will not be appropriate. If quantitative information is desired, numerical methods (see Chapter 13) may be necessary. In this subsection we briefly describe some qualitative properties that may be derived for Laplace's equation.

Mean value theorem. Our solution of Laplace's equation inside a circle, obtained in Sec 2.5.2 by the method of separation of variables, yields an important result. If we evaluate the temperature at the origin, $r = 0$, we discover from (2.5.45) that

$$u(0, \theta) = a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta;$$

the temperature there equals the average value of the temperature at the edges of the circle. This is called the **mean value property** for Laplace's equation. It holds in general in the following specific sense. Suppose that we wish to solve Laplace's equation in any region R (see Fig. 2.5.4). Consider *any* point P inside R and a circle of any radius r_0 (such that the circle is inside R). Let the temperature on the circle be $f(\theta)$, using polar coordinates centered at P . Our previous analysis still holds, and thus **the temperature at any point is the average of the temperature along any circle of radius r_0 (lying inside R) centered at that point.**

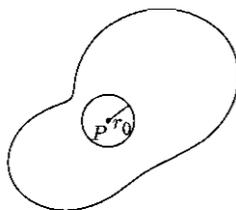


Figure 2.5.4: Circle within any region.

Maximum principles. We can use this to prove the **maximum principle** for Laplace's equation: **in steady state the temperature cannot attain its maximum in the interior** (unless the temperature is a constant everywhere) assuming no sources. The proof is by contradiction. Suppose that the maximum was at point P , as illustrated in Fig. 2.5.4. However, this should be the average of all points on any circle (consider the circle drawn). It is impossible for the temperature at P to be larger. This contradicts the original assumption, which thus cannot hold. We should not be surprised by the maximum principle. If the temperature was largest at point P , then in time the concentration of heat energy would diffuse and in steady state the maximum could not be in the interior. By letting $\psi = -u$, we can also show that the temperature cannot attain its minimum

in the interior. It follows that in **steady state the maximum and minimum temperatures occur on the boundary.**

Well-posedness and uniqueness. The maximum principle is a very important tool for further analysis of partial differential equations, especially in establishing qualitative properties (see, e.g., Protter and Weinberger [1967]). We say that a problem is **well posed** if there exists a unique solution that depends continuously on the nonhomogeneous data (i.e., the solution varies a small amount if the data are slightly changed). This is an important concept for physical problems. If the solution changed dramatically with only a small change in the data, then any physical measurement would have to be exact in order for the solution to be reliable. Fortunately, most standard problems in partial differential equations are well posed. For example, the maximum principle can be used to prove that Laplace's equation $\nabla^2 u = 0$ with u specified as $u = f(\mathbf{x})$ on the boundary is well posed. Suppose that we vary the boundary data a small amount such that

$$\nabla^2 v = 0 \quad \text{with} \quad v = g(\mathbf{x})$$

on the boundary, where $g(\mathbf{x})$ is nearly the same as $f(\mathbf{x})$ everywhere on the boundary. We consider the difference between these two solutions, $w = u - v$. Due to the linearity,

$$\nabla^2 w = 0 \quad \text{with} \quad w = f(\mathbf{x}) - g(\mathbf{x})$$

on the boundary. The maximum (and minimum) principles for Laplace's equation imply that the maximum and minimum occur on the boundary. Thus, at any point inside,

$$\min(f(\mathbf{x}) - g(\mathbf{x})) \leq w \leq \max(f(\mathbf{x}) - g(\mathbf{x})). \quad (2.5.60)$$

Since $g(\mathbf{x})$ is nearly the same as $f(\mathbf{x})$ everywhere, w is small, and thus the solution v is nearly the same as u ; the solution of Laplace's equation slightly varies if the boundary data are slightly altered.

We can also prove that the solution of Laplace's equation is unique. We prove this by contradiction. Suppose that there are two solutions, u and v as above, which satisfy the same boundary condition [i.e., let $(f(\mathbf{x}) = g(\mathbf{x}))$]. If we again consider the difference ($w = u - v$), then the maximum and minimum principles imply [see (2.5.60)] that inside the region

$$0 \leq w \leq 0.$$

We conclude that $w = 0$ everywhere inside, and thus $u = v$ proving that if a solution exists, it must be unique. These properties (uniqueness and continuous dependence on the data) show that Laplace's equation with u specified on the boundary is a well-posed problem.

Solvability condition. If on the boundary the heat flow $-K_0 \nabla u \cdot \hat{n}$ is specified instead of the temperature, Laplace's equation may have no solutions [for a one-dimensional example, see Exercise 1.4.7(b)]. To show this, we integrate $\nabla^2 u = 0$ over the entire two-dimensional region

$$0 = \iint \nabla^2 u \, dx \, dy = \iint \nabla \cdot (\nabla u) \, dx \, dy.$$

Using the (two-dimensional) divergence theorem, we conclude that (see Exercise 1.5.8)

$$0 = \oint \nabla u \cdot \hat{n} \, ds. \quad (2.5.61)$$

Since $\nabla u \cdot \hat{n}$ is proportional to the heat flow through the boundary, (2.5.61) implies that the *net* heat flow through the boundary must be zero in order for a steady state to exist. This is clear physically, because otherwise there would be a change (in time) of the thermal energy inside, violating the steady-state assumption. Equation (2.5.61) is called the **solvability condition** or **compatibility condition** for Laplace's equation.

EXERCISES 2.5

2.5.1. Solve Laplace's equation inside a rectangle $0 \leq x \leq L$, $0 \leq y \leq H$, with the following boundary conditions:

$$*(a) \quad \frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(L, y) = 0, \quad u(x, 0) = 0, \quad u(x, H) = f(x)$$

$$(b) \quad \frac{\partial u}{\partial x}(0, y) = g(y), \quad \frac{\partial u}{\partial x}(L, y) = 0, \quad u(x, 0) = 0, \quad u(x, H) = 0$$

$$*(c) \quad \frac{\partial u}{\partial x}(0, y) = 0, \quad u(L, y) = g(y), \quad u(x, 0) = 0, \quad u(x, H) = 0$$

$$(d) \quad u(0, y) = g(y), \quad u(L, y) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = 0, \quad u(x, H) = 0$$

$$*(e) \quad u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) - \frac{\partial u}{\partial y}(x, 0) = 0, \quad u(x, H) = f(x)$$

$$(f) \quad u(0, y) = f(y), \quad u(L, y) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, H) = 0$$

$$(g) \quad \frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(L, y) = 0, \quad u(x, 0) = \begin{cases} 0 & x > L/2 \\ 1 & x < L/2 \end{cases}, \quad \frac{\partial u}{\partial y}(x, H) = 0$$

2.5.2. Consider $u(x, y)$ satisfying Laplace's equation inside a rectangle ($0 < x < L$, $0 < y < H$) subject to the boundary conditions

$$\frac{\partial u}{\partial x}(0, y) = 0 \quad \frac{\partial u}{\partial y}(x, 0) = 0$$

$$\frac{\partial u}{\partial x}(L, y) = 0 \quad \frac{\partial u}{\partial y}(x, H) = f(x).$$

*(a) Without solving this problem, briefly explain the physical condition under which there is a solution to this problem.

(b) Solve this problem by the method of separation of variables. Show that the method works only under the condition of part (a).

(c) The solution [part (b)] has an arbitrary constant. Determine it by consideration of the time-dependent heat equation (1.5.11) subject to the initial condition $u(x, y, 0) = g(x, y)$.

***2.5.3.** Solve Laplace's equation *outside* a circular disk ($r \geq a$) subject to the boundary condition:

$$(a) \quad u(a, \theta) = \ln 2 + 4 \cos 3\theta \quad (b) \quad u(a, \theta) = f(\theta)$$

You may assume that $u(r, \theta)$ remains finite as $r \rightarrow \infty$.

***2.5.4.** For Laplace's equation inside a circular disk ($r \leq a$), using (2.5.45) and (2.5.47), show that

$$u(r, \theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\bar{\theta}) \left[-\frac{1}{2} + \sum_{n=0}^{\infty} \left(\frac{r}{a} \right)^n \cos n(\theta - \bar{\theta}) \right] d\bar{\theta}.$$

Using $\cos z = \operatorname{Re} [e^{iz}]$, sum the resulting geometric series to obtain Poisson's integral formula.

2.5.5. Solve Laplace's equation inside the quarter-circle of radius 1 ($0 \leq \theta \leq \pi/2$, $0 \leq r \leq 1$) subject to the boundary conditions:

$$*(a) \quad \frac{\partial u}{\partial \theta}(r, 0) = 0, \quad u(r, \frac{\pi}{2}) = 0, \quad u(1, \theta) = f(\theta)$$

$$(b) \quad \frac{\partial u}{\partial \theta}(r, 0) = 0, \quad \frac{\partial u}{\partial \theta}(r, \frac{\pi}{2}) = 0, \quad u(1, \theta) = f(\theta)$$

$$*(c) \quad u(r, 0) = 0, \quad u(r, \frac{\pi}{2}) = 0, \quad \frac{\partial u}{\partial r}(1, \theta) = f(\theta)$$

$$(d) \quad \frac{\partial u}{\partial \theta}(r, 0) = 0, \quad \frac{\partial u}{\partial \theta}(r, \frac{\pi}{2}) = 0, \quad \frac{\partial u}{\partial r}(1, \theta) = g(\theta)$$

Show that the solution [part (d)] exists only if $\int_0^{\pi/2} g(\theta) d\theta = 0$. Explain this condition physically.

2.5.6. Solve Laplace's equation inside a semicircle of radius a ($0 < r < a$, $0 < \theta < \pi$) subject to the boundary conditions:

*(a) $u = 0$ on the diameter and $u(a, \theta) = g(\theta)$.

(b) the diameter is insulated and $u(a, \theta) = g(\theta)$.

2.5.7. Solve Laplace's equation inside a 60° wedge of radius a subject to the boundary conditions:

$$(a) \quad u(r, 0) = 0, \quad u(r, \frac{\pi}{3}) = 0, \quad u(a, \theta) = f(\theta)$$

$$*(b) \quad \frac{\partial u}{\partial \theta}(r, 0) = 0, \quad \frac{\partial u}{\partial \theta}(r, \frac{\pi}{3}) = 0, \quad u(a, \theta) = f(\theta)$$

2.5.8. Solve Laplace's equation inside a circular annulus ($a < r < b$) subject to the boundary conditions:

- * (a) $u(a, \theta) = f(\theta), \quad u(b, \theta) = g(\theta)$
 (b) $\frac{\partial u}{\partial r}(a, \theta) = 0, \quad u(b, \theta) = g(\theta)$
 (c) $\frac{\partial u}{\partial r}(a, \theta) = f(\theta), \quad \frac{\partial u}{\partial r}(b, \theta) = g(\theta)$

If there is a solvability condition, state it and explain it physically.

***2.5.9.** Solve Laplace's equation inside a 90° sector of a circular annulus ($a < r < b$, $0 < \theta < \pi/2$) subject to the boundary conditions:

- (a) $u(r, 0) = 0, \quad u(r, \pi/2) = 0, \quad u(a, \theta) = 0, \quad u(b, \theta) = f(\theta)$
 (b) $u(r, 0) = 0, \quad u(r, \pi/2) = f(r), \quad u(a, \theta) = 0, \quad u(b, \theta) = 0$

2.5.10. Using the maximum principles for Laplace's equation, prove that the solution of Poisson's equation, $\nabla^2 u = g(\mathbf{x})$, subject to $u = f(\mathbf{x})$ on the boundary, is unique.

2.5.11. Do Exercise 1.5.8.

2.5.12. (a) Using the divergence theorem, determine an alternative expression for $\iint u \nabla^2 u \, dx \, dy \, dz$.

- (b) Using part (a), prove that the solution of Laplace's equation $\nabla^2 u = 0$ (with u given on the boundary) is unique.
 (c) Modify part (b) if $\nabla u \cdot \hat{\mathbf{n}} = 0$ on the boundary.
 (d) Modify part (b) if $\nabla u \cdot \hat{\mathbf{n}} + hu = 0$ on the boundary. Show that Newton's law of cooling corresponds to $h < 0$.

2.5.13. Prove that the temperature satisfying Laplace's equation cannot attain its minimum in the interior.

2.5.14. Show that the "backwards" heat equation

$$\frac{\partial u}{\partial t} = -k \frac{\partial^2 u}{\partial x^2}$$

subject to $u(0, t) = u(L, t) = 0$ and $u(x, 0) = f(x)$, is *not* well posed. [Hint: Show that if the data are changed an arbitrarily small amount, for example

$$f(x) \rightarrow f(x) + \frac{1}{n} \sin \frac{n\pi x}{L}$$

for large n , then the solution $u(x, t)$ changes by a large amount.]

2.5.15. Solve Laplace's equation inside a semi-infinite strip ($0 < x < \infty$, $0 < y < H$) subject to the boundary conditions:

- (a) $\frac{\partial u}{\partial y}(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, H) = 0, \quad u(0, y) = f(y)$
 (b) $u(x, 0) = 0, \quad u(x, H) = 0, \quad u(0, y) = f(y)$
 (c) $u(x, 0) = 0, \quad u(x, H) = 0, \quad \frac{\partial u}{\partial x}(0, y) = f(y)$
 (d) $\frac{\partial u}{\partial y}(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, H) = 0, \quad \frac{\partial u}{\partial x}(0, y) = f(y)$

Show that the solution [part (d)] exists only if $\int_0^H f(y) \, dy = 0$.

2.5.16. Consider Laplace's equation inside a rectangle $0 \leq x \leq L$, $0 \leq y \leq H$, with the boundary conditions

$$\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(L, y) = g(y), \quad \frac{\partial u}{\partial y}(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, H) = f(x).$$

- (a) What is the solvability condition and its physical interpretation?
 (b) Show that $u(x, y) = A(x^2 - y^2)$ is a solution if $f(x)$ and $g(y)$ are constants [under the conditions of part (a)].
 (c) Under the conditions of part (a), solve the general case [nonconstant $f(x)$ and $g(y)$]. [Hints: Use part (b) and the fact that $f(x) = f_{av} + [f(x) - f_{av}]$, where $f_{av} = \frac{1}{L} \int_0^L f(x) \, dx$.]

2.5.17. Show that the mass density $\rho(x, t)$ satisfies $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$ due to conservation of mass.

2.5.18. If the mass density is constant, using the result of 1 show that $\nabla \cdot \mathbf{u} = 0$

2.5.19. Show that the streamlines are parallel to the fluid velocity.

2.5.20. Show that anytime there is a streamfunction, $\nabla \times \mathbf{u} = 0$.

2.5.21. From $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$, derive $u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$, $u_\theta = -\frac{\partial \psi}{\partial r}$.

2.5.22. Show the drag force is zero for a uniform flow past a cylinder including circulation.

2.5.23. Consider the velocity u_θ at the cylinder. Where does the maximum and minimum occur?

2.5.24. Consider the velocity u_θ at the cylinder. If the circulation is negative show that the velocity will be larger above the cylinder than below.

2.5.25. A stagnation point is a place where $\mathbf{u} = 0$. For what values of the circulation does a stagnation point exist on the cylinder?

2.5.26. For what values of θ will $u_r = 0$ off the cylinder? For these θ , where (for what values of r) will $u_\theta = 0$ also?

2.5.27. Show that $\psi = \alpha \frac{\sin \theta}{r}$ satisfies Laplace's equation. Show the streamlines are circles. Graph the streamlines.