11.3.5. Consider

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x,t) & u(x,0) &= f(x) \\ \frac{\partial u}{\partial x}(0,t) &= A(t) & \frac{\partial u}{\partial x}(L,t) &= B(t). \end{aligned}$$

(a) Solve for the appropriate Green's function using the method of eigenfunction expansion.

(b) Approximate the Green's function of part (a). Under what conditions is your approximation valid?

(c) Solve for the appropriate Green's function using the infinite space Green's function.

(d) Approximate the Green's function of part (c). Under what conditions is your approximation valid?

(e) Solve for $u(x,t)$ in terms of the Green's function.

11.3.6. Determine the Green's function for the heat equation subject to zero boundary conditions at $x - 0$ and $x = L$ by applying the method of eigenfunction expansions directly to the defining differential equation. [*Hint*: The answer is given by (11.3.35).]

Chapter 12

The Method of Characteristics for Linear and Quasi-Linear Wave Equations

12.1 Introduction

In previous chapters we obtained certain results concerning the one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \tag{12.1.1}$$

subject to the initial conditions

$$u(x,0) = f(x) \tag{12.1.2}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x). \tag{12.1.3}$$

For a vibrating string with zero displacement at $x = 0$ and $x = L$ we obtained a somewhat complicated Fourier sine series solution by the method of separation of variables in Chapter 4:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(a_n \cos\frac{n\pi ct}{L} + b_n \sin\frac{n\pi ct}{L}\right). \tag{12.1.4}$$

Further analysis of this solution [see (4.4.14) and Exercises 4.4.7 and 4.4.8] shows that the solution can be represented as the sum of a forward and backward moving wave. In particular,

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x_0)\, dx_0, \tag{12.1.5}$$

where $f(x)$ and $g(x)$ are the odd periodic extensions of the functions given in (12.1.2) and (12.1.3). We also obtained (12.1.5) in Chapter 11 for the one-dimensional wave equation without boundaries, using the infinite space Green's function.

In this chapter we introduce the more powerful method of characteristics to solve the one-dimensional wave equation. We will show in general that $u(x,t) = F(x - ct) + G(x+ct)$, where F and G are arbitrary functions. We will show that (12.1.5) follows for infinite space problems. Then we will discuss modifications needed to solve semi-infinite and finite domain problems. In Section 12.6, the method of characteristics will be applied to quasi-linear partial differential equations. There shock waves will be introduced when characteristics intersect.

12.2 Characteristics For First-Order Wave Equations

12.2.1 Introduction

The one-dimensional wave equation can be rewritten as

$$\boxed{\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0.} \tag{12.2.1}$$

A short calculation shows that it can be "factored" in two ways:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = 0$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0,$$

since the mixed second-derivative terms vanish in both. If we let

$$\boxed{w = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}} \tag{12.2.2}$$

$$\boxed{v = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x},} \tag{12.2.3}$$

we see that the one-dimensional wave equation (involving second derivatives) yields two **first-order wave equations**:

$$\boxed{\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0} \tag{12.2.4}$$

$$\boxed{\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0.} \tag{12.2.5}$$

12.2.2 Method of Characteristics for First-Order Partial Differential Equations

We begin by discussing either one of these simple first-order partial differential equations:

$$\boxed{\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0.} \tag{12.2.6}$$

The methods we will develop will be helpful in analyzing the one-dimensional wave equation (12.2.1). We consider the rate of change of $w(x(t),t)$ as measured by a moving observer, $x = x(t)$. The chain rule[1] implies that

$$\boxed{\frac{d}{dt}w(x(t),t) = \frac{\partial w}{\partial t} + \frac{dx}{dt}\frac{\partial w}{\partial x}.} \tag{12.2.7}$$

The first term $\partial w/\partial t$ represents the change in w at the fixed position, while $(dx/dt)(\partial w/\partial x)$ represents the change due to the fact that the observer moves into a region of possibly different w. Compare (12.2.7) with the partial differential equation for w, equation (12.2.6). It is apparent that *if* the observer moves with velocity c, that is, if

$$\boxed{\frac{dx}{dt} = c,} \tag{12.2.8}$$

then

$$\boxed{\frac{dw}{dt} = 0.} \tag{12.2.9}$$

Thus, w is constant. An observer moving with this special speed c would measure no changes in w.

Characteristics. In this way, the partial differential equation (12.2.6) has been replaced by two ordinary differential equations, (12.2.8) and (12.2.9). Integrating (12.2.8) yields

$$\boxed{x = ct + x_0,} \tag{12.2.10}$$

[1] Here d/dt as measured by a moving observer is sometimes called the substantial derivative.

the equation for the family of parallel **characteristics**[2] of (12.2.6), sketched in Fig. 12.2.1. Note that at $t = 0$, $x = x_0$. $w(x,t)$ is constant *along this line* (not necessarily constant everywhere). w **propagates** as a **wave** with **wave speed c** [see (12.2.8)].

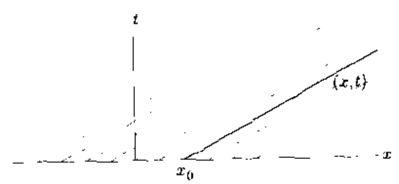

Figure 12.2.1: Characteristics for the first-order wave equation.

General solution. If $w(x,t)$ is given initially at $t = 0$,

$$w(x,0) = P(x), \tag{12.2.11}$$

then let us determine w at the point (x,t). Since w is constant along the characteristic,

$$w(x,t) = w(x_0, 0) = P(x_0).$$

Given x and t, the parameter is known from the characteristic, $x_0 = x - ct$, and thus

$$\boxed{w(x,t) = P(x - ct),} \tag{12.2.12}$$

which we call the general solution of (12.2.6).

We can think of $P(x)$ as being an arbitrary function. To verify this, we substitute (12.2.12) back into the partial differential equation (12.2.6). Using the chain rule

$$\frac{\partial w}{\partial x} = \frac{dP}{d(x-ct)} \frac{\partial (x-ct)}{\partial x} = \frac{dP}{d(x-ct)}$$

and

$$\frac{\partial w}{\partial t} = \frac{dP}{d(x-ct)} \frac{\partial (x-ct)}{\partial t} = -c\frac{dP}{d(x-ct)}.$$

Thus, it is verified that (12.2.6) is satisfied by (12.2.12). The general solution of a first-order partial differential equation contains an arbitrary function, while the general solution to ordinary differential equations contains arbitrary constants.

Example. Consider

$$\frac{\partial w}{\partial t} + 2\frac{\partial w}{\partial x} = 0,$$

subject to the initial condition

$$w(x,0) = \begin{cases} 0 & x < 0 \\ 4x & 0 < x < 1 \\ 0 & x > 1. \end{cases}$$

[2] A characteristic is a curve along which a PDE reduces to an ODE.

We have shown that w is constant along the characteristics $x - 2t =$ constant, keeping its same shape moving at velocity 2 (to the right). The important characteristics, $x = 2t + 0$ and $x = 2t + 1$, as well as a sketch of the solution at various times, appear in Fig. 12.2.2. $w(x,t) = 0$ if $x > 2t + 1$ or if $x < 2t$. Otherwise, by shifting

$$w(x,t) = 4(x - 2t) \qquad \text{if } 2t < x < 2t + 1.$$

To derive this analytic solution, we use the characteristic which starts at $x = x_0$:

$$x = 2t + x_0.$$

Along this characteristic, $w(x,t)$ is constant. If $0 < x_0 < 1$, then

$$w(x,t) = w(x_0, 0) = 4x_0 = 4(x - 2t),$$

as before. This is valid if $0 < x_0 < 1$ or equivalently $0 < x - 2t < 1$.

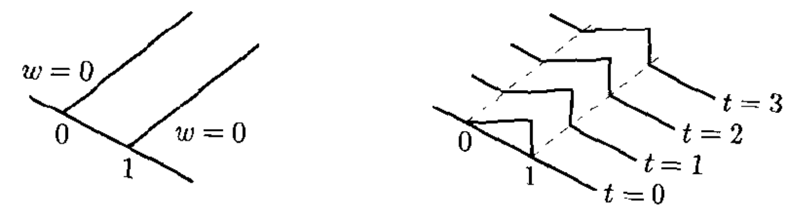

Figure 12.2.2: Propagation for the first-order wave equation.

Same shape. In general $w(x,t) = P(x - ct)$. *At fixed t, the solution of the first-order wave equation is the same shape shifted a distance ct* (distance = velocity times time). We illustrate this in Fig. 12.2.3.

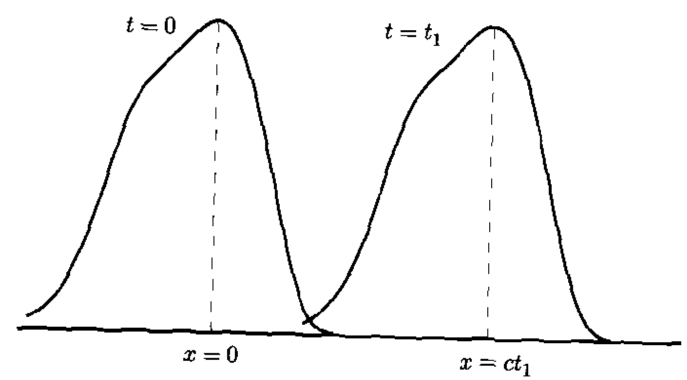

Figure 12.2.3: Shape invariance for the first-order wave equation.

Summary. The method of characteristics solves the first-order wave equation (12.2.6). In Sections 12.3-12.5, this method is applied to solve the wave equation (12.1.1). The reader may proceed directly to Section 12.6 where the method of characteristics is described for quasi-linear partial differential equations.

EXERCISES 12.2

12.2.1. Show that the wave equation can be considered as the following system of two coupled first-order partial differential equations:

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = w \qquad \frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0.$$

***12.2.2.** Solve

$$\frac{\partial w}{\partial t} - 3\frac{\partial w}{\partial x} = 0 \quad \text{with } w(x,0) = \cos x.$$

12.2.3. Solve

$$\frac{\partial w}{\partial t} + 4\frac{\partial w}{\partial x} = 0 \quad \text{with } w(0,t) = \sin 3t.$$

12.2.4. Solve

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0 \quad (c > 0) \quad \text{for } x > 0 \text{ and } t > 0 \text{ if}$$

$$w(x,0) = f(x) \quad x > 0 \qquad w(0,t) = h(t) \quad t > 0.$$

12.2.5. Solve using the method of characteristics (if necessary, see Section 12.6):

(a) $\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = e^{2x}$ with $w(x,0) = f(x)$

*(b) $\frac{\partial w}{\partial t} + x\frac{\partial w}{\partial x} = 1$ with $w(x,0) = f(x)$

(c) $\frac{\partial w}{\partial t} + t\frac{\partial w}{\partial x} = 1$ with $w(x,0) = f(x)$

*(d) $\frac{\partial w}{\partial t} + 3t\frac{\partial w}{\partial x} = w$ with $w(x,0) = f(x)$

***12.2.6.** Consider (if necessary, see Section 12.6):

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0 \quad \text{with} \quad u(x,0) = f(x).$$

Show that the characteristics are straight lines.

12.2.7. Consider Exercise 12.2.6 with

$$u(x,0) = f(x) = \begin{cases} 1 & x < 0 \\ 1 + x/L & 0 < x < L \\ 2 & x > L. \end{cases}$$

(a) Determine equations for the characteristics. Sketch the characteristics.

(b) Determine the solution $u(x,t)$. Sketch $u(x,t)$ for t fixed.

***12.2.8.** Consider Exercise 12.2.6 with

$$u(x,0) = f(x) = \begin{cases} 1 & x < 0 \\ 2 & x > 0. \end{cases}$$

Obtain the solution $u(x,t)$ by considering the limit as $L \to 0$ of the characteristics obtained in Exercise 12.2.7. Sketch characteristics and $u(x,t)$ for t fixed.

12.2.9. As motivated by the analysis of a moving observer, make a change of independent variables from (x,t) to a coordinate system moving with velocity c, (ξ, t'), where $\xi = x - ct$ and $t' = t$, in order to solve (12.2.6).

12.2.10. For the first-order "quasi-linear" partial differential equation

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c,$$

where a, b, and c are functions of x, y and u, show that the method of characteristics (if necessary, see Section 12.6) yields

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}.$$

12.3 Method of Characteristics for the One-Dimensional Wave Equation

12.3.1 Introduction

From the one-dimensional wave equation,

$$\boxed{\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0,} \qquad (12.3.1)$$

we derived two first-order partial differential equations, $\partial w/\partial t + c\partial w/\partial x = 0$ and $\partial v/\partial t - c\partial v/\partial x = 0$, where $u = \partial u/\partial t - c\partial u/\partial x$ and $v = \partial u/\partial t + c\partial u/\partial x$. We have shown that w remains the same shape moving at velocity c:

$$w = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = P(x - ct). \qquad (12.3.2)$$

The problem for v is identical (replace c by $-c$). Thus, we could have shown that v is translated unchanged at velocity $-c$:

$$v = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = Q(x + ct). \qquad (12.3.3)$$

By combining (12.3.2) and (12.3.3) we obtain, for example,

$$\frac{\partial u}{\partial t} = \frac{1}{2}[P(x - ct) + Q(x + ct)],$$

and thus

$$\boxed{u(x,t) = F(x - ct) + G(x + ct),} \qquad (12.3.4)$$

where F and G are arbitrary functions ($-cF' = \frac{1}{2}P$ and $cG' = \frac{1}{2}Q$). This result was obtained by d'Alembert in 1747.

The general solution is the sum of $F(x-ct)$, a wave of fixed shape moving to the right with velocity c, and $G(x+ct)$, a wave of fixed shape moving to the left with velocity $-c$. The solution may be sketched if $F(x)$ and $G(x)$ are known. We shift $F(x)$ to the right a distance ct and shift $G(x)$ to the left a distance ct and add the two. Although each shape is unchanged, the sum will in general be a shape that is changing in time. In Sec. 12.3.2 we will show how to determine $F(x)$ and $G(x)$ from initial conditions.

Characteristics. Part of the solution is constant along the family of characteristics $x - ct =$ constant, while a different part of the solution is constant along $x + ct =$ constant. For the one-dimensional wave equation, (12.3.1), there are two families of characteristic curves, as sketched in Fig. 12.3.1.

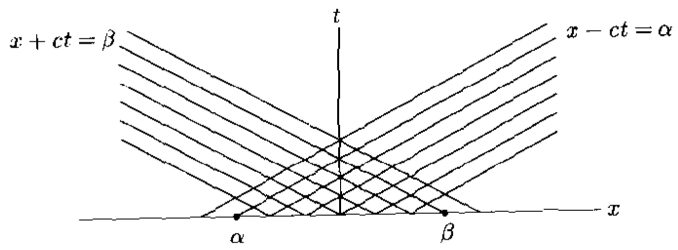

Figure 12.3.1: Characteristics for the one-dimensional wave equation.

12.3.2 Initial Value Problem (Infinite Domain)

In Sec. 12.3.1 we showed that the general solution of the one-dimensional wave equation is

$$\boxed{u(x,t) = F(x-ct) + G(x+ct).} \tag{12.3.5}$$

Here we will determine the arbitrary functions in order to satisfy the initial conditions:

$$\boxed{u(x,0) = f(x)} \quad -\infty < x < \infty \tag{12.3.6}$$

$$\boxed{\frac{\partial u}{\partial t}(x,0) = g(x)} \quad -\infty < x < \infty. \tag{12.3.7}$$

These initial conditions imply that

$$f(x) = F(x) + G(x) \tag{12.3.8}$$

$$\frac{g(x)}{c} = -\frac{dF}{dx} + \frac{dG}{dx}. \tag{12.3.9}$$

We solve for $G(x)$ by eliminating $F(x)$; for example, adding the derivative of (12.3.8) to (12.3.9) yields

$$\frac{dG}{dx} = \frac{1}{2}\left(\frac{df}{dx} + \frac{g(x)}{c}\right).$$

By integrating this, we obtain

$$\boxed{G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\overline{x})\, d\overline{x}} + k \tag{12.3.10}$$

$$\boxed{F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\overline{x})\, d\overline{x}} - k, \tag{12.3.11}$$

where the latter equation was obtained from (12.3.8). k can be neglected since $u(x,t)$ is obtained from (12.3.5) by adding (12.3.10) and (12.3.11) (with appropriate shifts).

Sketching technique. The solution $u(x,t)$ can be graphed based on (12.3.5) in the following straightforward manner:

1. Given $f(x)$ and $g(x)$. Obtain the graphs of

$$\frac{1}{2}f(x) \quad \text{and} \quad \frac{1}{2c}\int_0^x g(\overline{x})\, d\overline{x},$$

   the latter by integrating first.
2. By addition and subtraction, from $F(x)$ and $G(x)$; see (12.3.10) and (12.3.11).
3. Translate (shift) $F(x)$ to the right a distance ct and $G(x)$ to the left ct.
4. Add the two shifted functions, thus satisfying (12.3.5).

Initially at rest. If a vibrating string is initially at rest [$\partial u/\partial t(x,0) = g(x) = 0$], then from (12.3.10) and (12.3.11) $F(x) = G(x) = \frac{1}{2}f(x)$. Thus,

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)]. \tag{12.3.12}$$

The initial condition $u(x,0) = f(x)$ splits into two parts; half moves to the left and half to the right.

Example. Suppose that an infinite vibrating string is initially stretched into the shape of a single rectangular pulse and is let go from rest. The corresponding initial conditions are

$$u(x,0) = f(x) = \begin{cases} 1 & |x| < h \\ 0 & |x| > h. \end{cases}$$

and

$$\frac{\partial u}{\partial t}(x,0) = g(x) = 0.$$

The solution is given by (12.3.12). By adding together these two rectangular pulses, we obtain Fig. 12.3.2. The pulses overlap until the left end of the right-moving one passes the right end of the other. Since each is traveling at speed c, they are moving

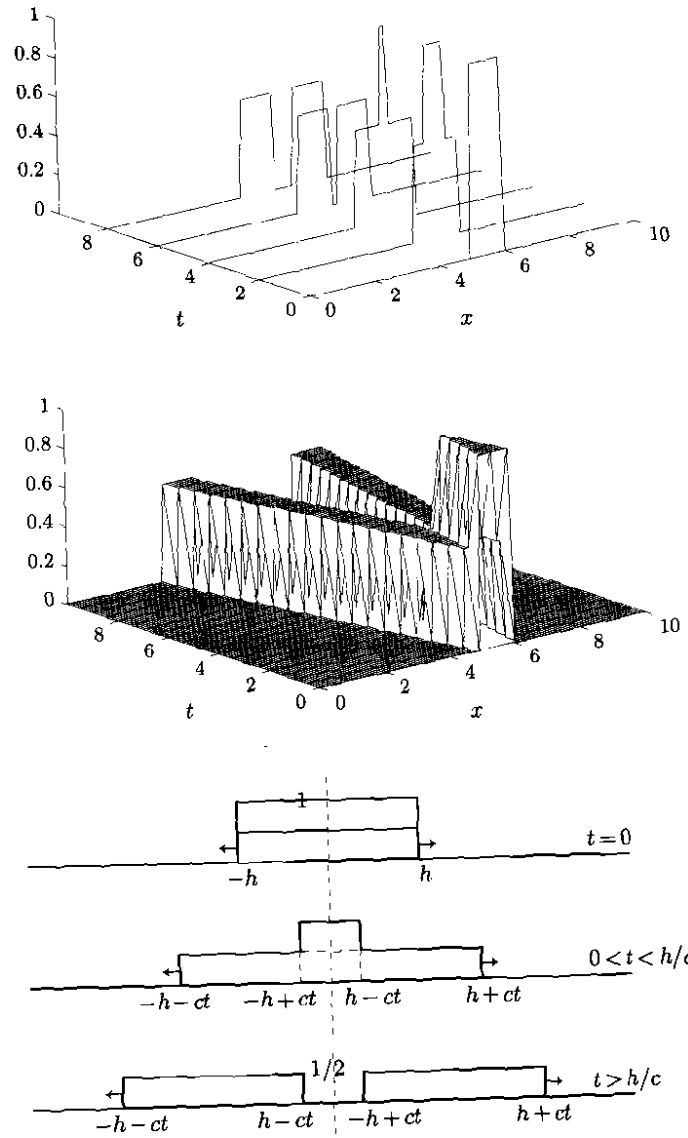

Figure 12.3.2: Initial value problem for the one-dimensional wave equation.

apart at velocity $2c$. The ends are initially a distance $2h$ apart, and hence the time at which the two pulses separate is

$$t = \frac{\text{distance}}{\text{velocity}} = \frac{2h}{2c} = \frac{h}{c}.$$

Important characteristics are sketched in Fig. 12.3.3. F stays constant moving to the right at velocity c, while G stays constant moving to the left. From (12.3.10) and (12.3.11)

$$F(x) = G(x) = \begin{cases} \frac{1}{2} & |x| < h \\ 0 & |x| > h. \end{cases}$$

This information also appears in Fig. 12.3.3.

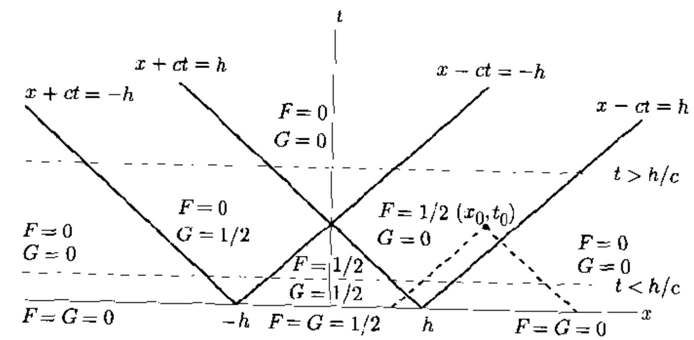

Figure 12.3.3: Method of characteristics for the one-dimensional wave equation.

Example not at rest. Suppose that an infinite string is initially horizontally stretched with prescribed initial velocity as follows:

$$\begin{aligned} u(x,0) = f(x) &= 0 \\ \frac{\partial u}{\partial t}(x,0) = g(x) &= \begin{cases} 1 & |x| < h \\ 0 & |x| > h. \end{cases} \end{aligned}$$

In Exercise 12.3.2 it is shown that this corresponds to instantaneously applying a constant impulsive force to the entire region $|x| < h$, as though the string is being struck by a broad ($|x| < h$) hammer. The calculation of the solution of the wave equation with these initial conditions is more involved than in the preceding example. From (12.3.10) and (12.3.11), we need $\int_0^x g(\overline{x})\, d\overline{x}$, representing the area under $g(x)$ from 0 to x:

$$2cG(x) = -2cF(x) = \int_0^x g(\overline{x})d\overline{x} = \begin{cases} -h & x < -h \\ x & -h < x < h \\ h & x > h. \end{cases}$$

The solution $u(x,t)$ is the sum of $F(x)$ shifted to the right (at velocity c) and $G(x)$ shifted to the left (at velocity c). $F(x)$ and $G(x)$ are sketched in Fig. 12.3.4, as is

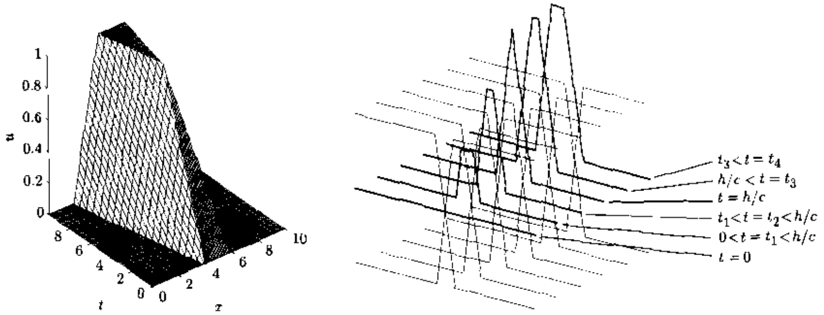

Figure 12.3.4: Time evolution for a struck string.

their shifted sum. The striking of the broad hammer causes the displacement of the string to gradually increase near where the hammer hit and to have this disturbance spread out to the left and right as time increases. Eventually, the string reaches an elevated rest position. Alternatively, the solution can be obtained in an algebraic way (see Exercise 12.3.5). The characteristics sketched in Fig. 12.3.3 are helpful.

12.3.3 d'Alembert's Solution

The general solution of the one-dimensional wave equation can be simplified somewhat. Substituting (12.3.10) and (12.3.11) into the general solution (12.3.5) yields

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\left[\int_0^{x+ct} g(\overline{x})\,d\overline{x} - \int_0^{x-ct} g(\overline{x})\,d\overline{x}\right]$$

or

$$\boxed{u(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\overline{x})\,d\overline{x},} \qquad (12.3.13)$$

known as **d'Alembert's solution** (previously obtained by Fourier transform methods). It is a very elegant result. However, for sketching solutions often it is easier to work directly with (12.3.10) and (12.3.11), where these are shifted according to (12.3.5).

Domain of dependence and range of influence. The importance of the characteristics $x - ct =$ constant and $x + ct =$ constant is clear. At position x at time t the initial position data are needed at $x \pm ct$, while all the initial velocity data between $x - ct$ and $x + ct$ is needed. The region between $x - ct$ and $x + ct$ is called the **domain of dependence** of the solution at (x, t) as sketched in Fig. 12.3.5. In addition, we sketch the **range of influence**, the region affected by the initial data at one point.

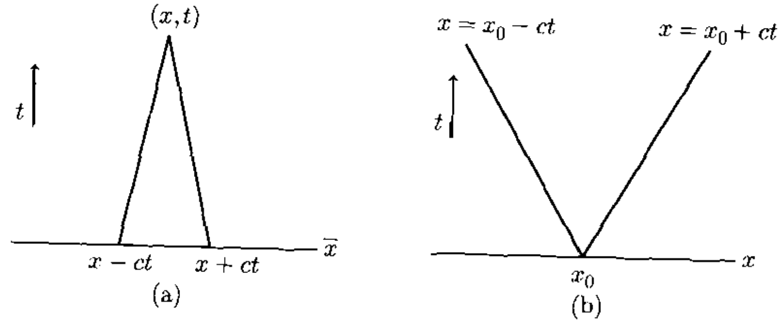

Figure 12.3.5: (a) Domain of dependence; (b) range of influence.

EXERCISES 12.3

12.3.1. Suppose that $u(x,t) = F(x-ct) + G(x+ct)$, where F and G are sketched in Fig. 12.3.6. Sketch the solution for various times.

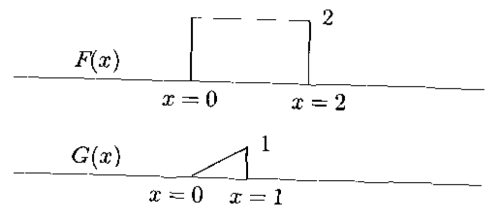

Figure 12.3.6

12.3.2. Suppose that a stretched string is unperturbed (horizontal, $u = 0$) and at rest ($\partial u/\partial t = 0$). If an impulsive force is applied at $t = 0$, the initial value problem is

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c\frac{\partial^2 u}{\partial x^2} + \alpha(x)\delta(t) \\ u(x,t) &= 0 \qquad t < 0. \end{aligned}$$

(a) Without using explicit solutions, show that this is equivalent to

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad t > 0$$

subject to $u(x,0) = 0$ and $\frac{\partial u}{\partial t}(x,0) = \alpha(x)$.

Thus, the initial velocity $\alpha(x)$ is equivalent to an impulsive force.

(b) Do part (a) using the explicit solution of both problems.

12.3.3. An alternative way to solve the one-dimensional wave equation (12.3.1) is based on (12.3.2) and (12.3.3). Solve the wave equation by introducing a change of variables from (x,t) to two moving coordinates (ξ, η) one moving to the left (with velocity $-c$) and one moving to the right (with velocity c):

$$\xi = x - ct \qquad \text{and} \qquad \eta = x + ct.$$

***12.3.4.** Suppose that $u(x,t) = F(x - ct)$. Evaluate

(a) $\frac{\partial u}{\partial t}(x,0)$ (b) $\frac{\partial u}{\partial x}(0,t)$

12.3.5. Determine analytic formulas for $u(x,t)$ if

$$\begin{aligned} u(x,0) = f(x) &= 0 \\ \frac{\partial u}{\partial t}(x,0) = g(x) &= \begin{cases} 1 & |x| < h \\ 0 & |x| > h. \end{cases} \end{aligned}$$

(*Hint*: Using characteristics as sketched in Fig. 12.3.3, show there are two distinct regions $t < h/c$ and $t > h/c$. In each, show that the solution has five different forms, depending on x.)

12.3.6. Consider the three-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u.$$

Assume that the solution is **spherically symmetric**, so that

$$\nabla^2 u = (1/\rho^2)(\partial/\partial\rho)(\rho^2 \partial u/\partial\rho).$$

(a) Make the transformation $u = (1/\rho)w(\rho,t)$ and verify that

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial \rho^2}.$$

(b) Show that the most general spherically symmetric solution of the wave equation consists of the sum of two spherically symmetric waves, one moving outward at speed c and the other inward at speed c. Note the decay of the amplitude.

12.4 Semi-Infinite Strings and Reflections

We will solve the one-dimensional wave equation on a semi-infinite interval, $x > 0$:

$$\text{PDE:} \quad \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}} \tag{12.4.1}$$

$$\text{IC1:} \quad \boxed{u(x,0) = f(x)} \tag{12.4.2}$$

$$\text{IC2:} \quad \boxed{\frac{\partial u}{\partial t}(x,0) = g(x).} \tag{12.4.3}$$

A condition is necessary at the boundary $x = 0$. We suppose that the string is fixed at $x = 0$:

$$\text{BC:} \quad \boxed{u(0,t) = 0.} \tag{12.4.4}$$

Although a Fourier sine transform can be used, we prefer to indicate how to use the general solution and the method of characteristics:

$$\boxed{u(x,t) = F(x - ct) + G(x + ct).} \tag{12.4.5}$$

As in Sec. 12.3, the initial conditions are satisfied if

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\overline{x})\, d\overline{x} \qquad x > 0 \tag{12.4.6}$$

$$G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\overline{x})\, d\overline{x} \qquad x > 0. \tag{12.4.7}$$

However, it is very important to note that (unlike the case of the infinite string) (12.4.6) and (12.4.7) are valid only for $x > 0$; the arbitrary functions are only determined from the initial conditions for positive arguments. In the general solution, $G(x + ct)$ requires only positive arguments of G (since $x > 0$ and $t > 0$). On the other hand, $F(x - ct)$ requires positive arguments if $x > ct$ but requires negative arguments if $x < ct$. As indicated by a space-time diagram, Fig. 12.4.1, the information that there is a fixed end at $x = 0$ travels at a finite velocity c. Thus, if $x > ct$, the string does not know that there is any boundary. In this case ($x > ct$), the solution is obtained as before [using (12.4.6) and (12.4.7)],

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\overline{x})\, d\overline{x}, \qquad x > ct, \tag{12.4.8}$$

d'Alembert's solution. However, here this is not valid if $x < ct$. Since $x + ct > 0$,

$$G(x+ct) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(\overline{x})\, d\overline{x},$$

as determined earlier. To obtain F for negative arguments, we cannot use the initial conditions. Instead, the boundary condition must be utilized. $u(0,t) = 0$ implies

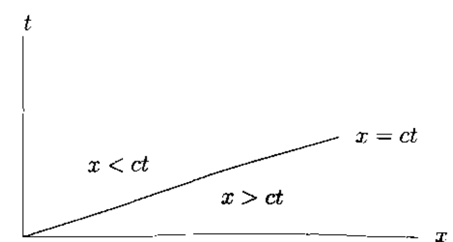

Figure 12.4.1: Characteristic emanating from the boundary.

that [from (12.4.5)]

$$0 = F(-ct) + G(ct) \qquad \text{for } t > 0. \tag{12.4.9}$$

Thus, F for negative arguments is $-G$ of the corresponding positive argument:

$$F(z) = -G(-z) \qquad \text{for } z < 0. \tag{12.4.10}$$

Thus, the solution for $x - ct < 0$ is

$$\begin{aligned} u(x,t) &= F(x-ct) + G(x+ct) = G(x+ct) - G(ct-x) \\ &= \frac{1}{2}[f(x+ct) - f(ct-x)] + \frac{1}{2c}\left[\int_0^{x+ct} g(\overline{x})\,d\overline{x} - \int_0^{ct-x} g(\overline{x})\,d\overline{x}\right] \\ &= \frac{1}{2}[f(x+ct) - f(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\overline{x})\,d\overline{x}. \end{aligned}$$

To interpret this solution, the method of characteristics is helpful. Recall that for infinite problems $u(x,t)$ is the sum of F (moving to the right) and G (moving to the left). For semi-infinite problems with $x > ct$, the boundary does not affect the characteristics (see Fig. 12.4.2). If $x < ct$, then Fig. 12.4.3 shows the left-moving characteristic (G constant) not affected by the boundary, but the right-moving characteristic emanates from the boundary. F is constant moving to the right. Due to the boundary condition, $F + G = 0$ at $x = 0$, the right-moving wave is minus the left-moving wave. The wave inverts as it "bounces off" the boundary. The resulting right-moving wave $-G(ct - x)$ is called the **reflected wave**. For $x < ct$, the total solution is the reflected wave plus the as yet unreflected left-moving wave:

$$u(x,t) = G(x+ct) - G(-(x-ct)).$$

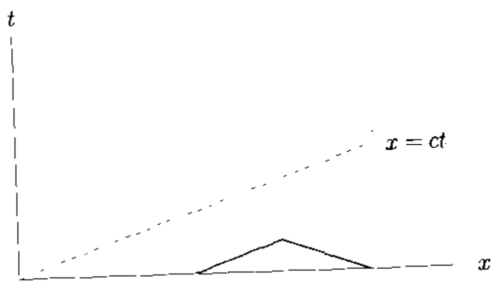

Figure 12.4.2: Characteristics.

The negatively reflected wave $-G(-(x-ct))$ moves to the right. It behaves *as if* initially at $t = 0$ it were $-G(-x)$. If there were no boundary, the right-moving wave $F(x - ct)$ would be initially $F(x)$. Thus, the reflected wave is exactly the wave that would have occurred if

$$F(x) = -G(-x) \qquad \text{for } x < 0,$$

or equivalently

$$\frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\overline{x})\,d\overline{x} = -\frac{1}{2}f(-x) - \frac{1}{2c}\int_0^{-x} g(\overline{x})\,d\overline{x}.$$

One way to obtain this is to extend the initial position $f(x)$ for $x > 0$ as an odd function [such that $f(-x) = -f(x)$] and also extend the initial velocity $g(x)$ for $x > 0$ as an odd function [then its integral, $\int_0^x g(\overline{x})\,d\overline{x}$, will be an even function]. *In summary,* **the solution of the semi-infinite problem with $u = 0$ at $x = 0$ is the same as an infinite problem with the initial positions and velocities extended as odd functions.**

As further explanation, suppose that $u(x,t)$ is any solution of the wave equation. Since the wave equation is unchanged when x is replaced by $-x$, $u(-x,t)$ (and any multiple of it) is also a solution of the wave equation. If the initial conditions satisfied by $u(x,t)$ are odd functions of x, then both $u(x,t)$ and $-u(-x,t)$ solve these initial conditions and the wave equation. Since the initial value problem has a unique solution, $u(x,t) = -u(-x,t)$; that is, $u(x,t)$, which is odd initially, will remain odd for all time. Thus, odd initial conditions yield a solution that will satisfy a zero boundary condition at $x = 0$.

Example. Consider a semi-infinite string $x > 0$ with a fixed end $u(0,t) = 0$, which is initially at rest, $\partial u/\partial t(x,0) = 0$, with an initial unit rectangular pulse,

$$f(x) = \begin{cases} 1 & 4 < x < 5 \\ 0 & \text{otherwise.} \end{cases}$$

Since $g(x) = 0$, it follows that

$$F(x) = G(x) = \frac{1}{2}f(x) = \begin{cases} \frac{1}{2} & 4 < x < 5 \\ 0 & \text{otherwise (with } x > 0). \end{cases}$$

F moves to the right; G moves to the left, negatively reflecting off $x = 0$. This can also be interpreted as an initial condition (on an infinite domain) with $f(x)$ and $g(x)$ extended as odd functions. The solution is sketched in Fig. 12.4.4. Note the *negative* reflection.

Problems with nonhomogeneous boundary conditions at $x = 0$ can be analyzed in a similar way.

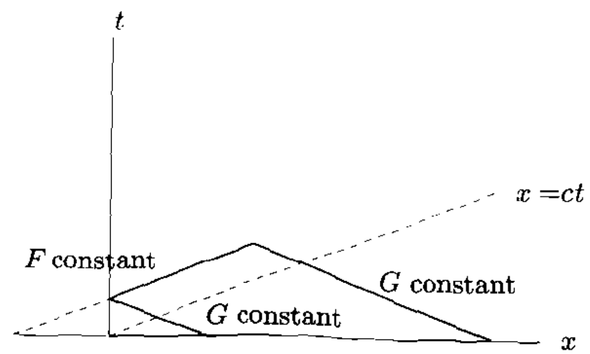

Figure 12.4.3: Reflected characteristics.

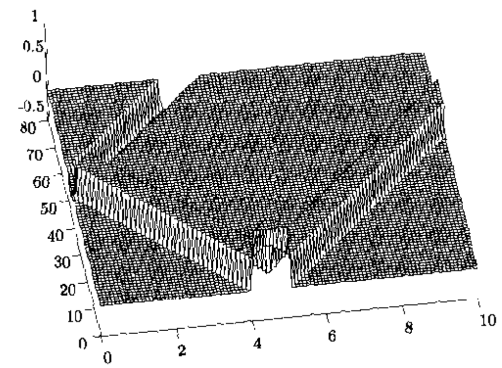

Figure 12.4.4: Reflected pulse.

EXERCISES 12.4

***12.4.1.** Solve by the method of characteristics:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad x > 0$$

subject to $u(x,0) = 0$, $\frac{\partial u}{\partial t}(x,0) = 0$, and $u(0,t) = h(t)$.

***12.4.2.** Determine $u(x,t)$ if

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{for } x < 0 \text{ only,}$$

where $u(x,0) = \cos x \quad x < 0$, $\frac{\partial u}{\partial t}(x,0) = 0 \quad x < 0$, $u(0,t) = e^{-t} \quad t > 0$. Do not sketch the solution. However, draw a space-time diagram, including all important characteristics.

12.4.3. Consider the wave equation on a semi-infinite interval

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 < x < \infty$$

with the free boundary condition $\frac{\partial u}{\partial x}(0,t) = 0$ and the initial conditions

$$u(x,0) = \begin{cases} 0 & 0 < x < 2 \\ 1 & 2 < x < 3 \\ 0 & x > 3 \end{cases} \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

Determine the solution. Sketch the solution for various times. (Assume that u is continuous at $x = 0$, $t = 0$.)

12.4.4. (a) Solve for $x > 0$, $t > 0$ (using the method of characteristics)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\left.\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x) \end{aligned}\right\} \quad x > 0$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad t > 0.$$

(Assume that u is continuous at $x = 0$, $t = 0$.)

(b) Show that the solution of part (a) may be obtained by extending the initial position and velocity as even functions (around $x = 0$).

(c) Sketch the solution if $g(x) = 0$ and

$$f(x) = \begin{cases} 1 & 4 < x < 5 \\ 0 & \text{otherwise.} \end{cases}$$

12.4.5. (a) Show that if $u(x,t)$ and $\partial u/\partial t$ are initially even around $x = x_0$, $u(x,t)$ will remain even for all time.

(b) Show that this type of even initial condition yields a solution that will satisfy a zero derivative boundary condition at $x = x_0$.

***12.4.6.** Solve ($x > 0, t > 0$)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

subject to the conditions $u(x,0) = 0$, $\frac{\partial u}{\partial t}(x,0) = 0$, and $\frac{\partial u}{\partial x}(0,t) = h(t)$.

***12.4.7.** Solve

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \begin{matrix} x > 0 \\ t > 0 \end{matrix}$$

subject to $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = 0$, and $\frac{\partial u}{\partial x}(0,t) = h(t)$. (Assume that u is continuous at $x = 0$, $t = 0$.)

12.4.8. Solve

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \text{ with } u(x,0) = 0 \text{ and } \frac{\partial u}{\partial t}(x,0) = 0,$$

subject to $u(x,t) = g(t)$ along $x = \frac{c}{2}t \, (c > 0)$.

12.5 Method of Characteristics for a Vibrating String of Fixed Length

In Chapter 2 we solved for the vibration of a finite string satisfying

$$\text{PDE:} \quad \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}} \qquad (12.5.1)$$

$$\text{BC:}\quad \boxed{\begin{array}{l} u(0,t)=0 \\ u(L,t)=0 \end{array}} \tag{12.5.2}$$

$$\text{IC:}\quad \boxed{\begin{array}{l} u(x,0)=f(x) \\ \dfrac{\partial u}{\partial t}(x,0)=g(x), \end{array}} \tag{12.5.3}$$

using Fourier series methods. We can obtain an equivalent, but in some ways more useful, result by using the general solution of the one-dimensional wave equation:

$$u(x,t) = F(x-ct) + G(x+ct). \tag{12.5.4}$$

The initial conditions are prescribed only for $0 < x < L$, and hence the formulas for $F(x)$ and $G(x)$ previously obtained are valid only for $0 < x < L$:

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\overline{x})\, d\overline{x} \tag{12.5.5}$$

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\overline{x})\, d\overline{x}. \tag{12.5.6}$$

If $0 < x - ct < L$ and $0 < x + ct < L$ as shown in Fig. 12.5.1, then d'Alembert's solution is valid:

$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\overline{x})\, d\overline{x}. \tag{12.5.7}$$

In this region the string does not know that either boundary exists; the information that there is a boundary propagates at velocity c from $x = 0$ and $x = L$.

Figure 12.5.1: Characteristics.

If one's position and time is such that signals from the boundary have already arrived, then modifications in (12.5.7) must be made. The boundary condition at $x = 0$ implies that

$$0 = F(-ct) + G(ct) \quad \text{for } t > 0, \tag{12.5.8}$$

while at $x = L$ we have

$$0 = F(L-ct) + G(L+ct) \quad \text{for } t > 0. \tag{12.5.9}$$

These in turn imply reflections and multiple reflections, as illustrated in Fig. 12.5.2.

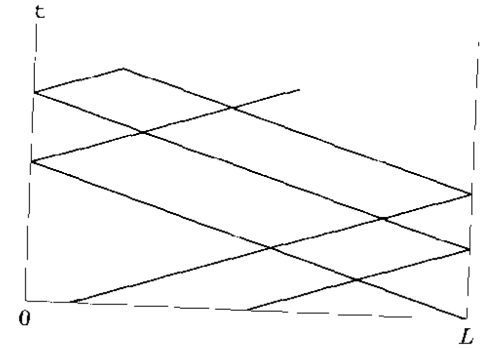

Figure 12.5.2: Multiply reflected characteristics.

Alternatively, a solution on an infinite domain without boundaries can be considered which is odd around $x = 0$ and odd around $x = L$, as sketched in Fig. 12.5.3. In this way, the zero condition at both $x = 0$ and $x = L$ will be satisfied. We note that $u(x,t)$ is periodic with period $2L$. In fact, we ignore the oddness around $x = L$, since periodic functions that are odd around $x = 0$ are automatically odd around $x = L$. Thus, the simplest way to obtain the solution is to **extend the initial conditions as odd functions (around $x = 0$) which are periodic (with period $2L$)**. With these odd periodic initial conditions, the method of characteristics can be utilized as well as d'Alembert's solution (12.5.7).

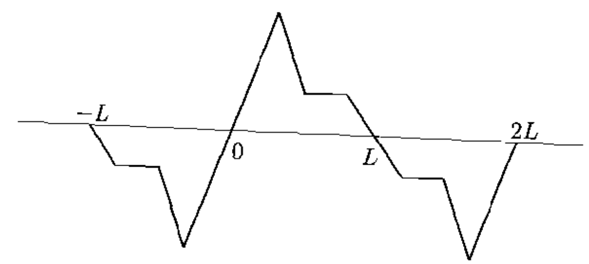

Figure 12.5.3: Odd periodic extension.

Example. Suppose that a string is initially at rest with prescribed initial conditions $u(x,0) = f(x)$. The string is fixed at $x = 0$ and $x = L$. Instead of using Fourier series methods, we extend the initial conditions as odd functions around $x = 0$ and $x = L$. Equivalently, we introduce the *odd periodic extension*. (The odd periodic extension is also used in the Fourier series solution.) Since the string is initially at rest, $g(x) = 0$, the odd periodic extension is $g(x) = 0$ for all x. Thus, the solution of the one-dimensional wave equation is the sum of two simple waves:

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)],$$

where $f(x)$ is the odd periodic extension of the given initial position. This solution is much simpler than the summation of the first 100 terms of its Fourier sine series.

EXERCISES 12.5

12.5.1. Consider

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \left.\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x) \end{aligned}\right\} \quad 0 < x < L$$

$$u(0,t) = 0 \qquad u(L,t) = 0.$$

(a) Obtain the solution by Fourier series techniques.

*(b) If $g(x) = 0$, show that part (a) is equivalent to the results of Chapter 12.

(c) If $f(x) = 0$, show that part (a) is equivalent to the results of Chapter 12.

12.5.2. Solve using the method of characteristics:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = 0 \qquad u(0,t) = h(t) \qquad \frac{\partial u}{\partial t}(x,0) = 0 \qquad u(L,t) = 0.$$

12.5.3. Consider

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 10$$

$$u(x,0) = f(x) = \begin{cases} 1 & 4 < x < 5 \\ 0 & \text{otherwise} \end{cases} \qquad u(0,t) = 0$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) = 0 \qquad \frac{\partial u}{\partial x}(L,t) = 0.$$

(a) Sketch the solution using the method of characteristics.

(b) Obtain the solution using Fourier - series - type techniques.

(c) Obtain the solution by converting to an equivalent problem on an infinite domain.

12.5.4. How should initial conditions be extended if $\partial u/\partial x(0,t) = 0$ and $u(L,t) = 0$?

12.6 The Method of Characteristics for Quasi-linear Partial Differential Equations

12.6.1 Method of Characteristics

Most of this text describes methods for solving linear partial differential equations (separation of variables, eigenfunction expansions, Fourier and Laplace transforms, Green's functions) that cannot be extended to nonlinear problems. However, the

method of characteristics, used to solve the wave equation, can be applied to partial differential equations of the form

$$\boxed{\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = Q,} \tag{12.6.1}$$

where c and Q may be functions of x, t, and ρ. When the coefficient c depends on the unknown solution ρ, (12.6.1) is not linear. Superposition is not valid. Nonetheless (12.6.1) is called a **quasi-linear** partial differential equation, since it is linear in the first partial derivatives, $\partial\rho/\partial t$ and $\partial\rho/\partial x$. To solve (12.6.1), we again consider an observer moving in some prescribed way $x(t)$. By comparing (12.2.7) and (12.6.1), we obtain

$$\boxed{\frac{d\rho}{dt} = Q(\rho, x, t),} \tag{12.6.2}$$

if

$$\boxed{\frac{dx}{dt} = c(\rho, x, t),} \tag{12.6.3}$$

The partial differential equation (12.6.1) reduces to two coupled ordinary differential equations along the special trajectory or direction defined by (12.6.3), known as a **characteristic curve**, or simply a **characteristic** for short. The velocity defined by (12.6.3) is called the **characteristic velocity**, or **local wave velocity**. A characteristic starting from $x = x_0$, as illustrated in Fig. 12.6.1, is determined from the coupled differential equations (12.6.2) and (12.6.3) using the initial conditions $\rho(x,0) = f(x)$. Along the characteristic, the solution ρ changes according to (12.6.2). Other initial positions yield other characteristics, generating a family of characteristics.

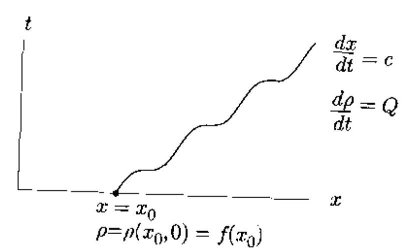

Figure 12.6.1: Characteristic starting from $x = x_0$ at time $t = 0$.

Example. If the local wave velocity c is a constant c_0 and $Q = 0$, then the quasi-linear partial differential equation (12.6.1) becomes the linear one, (12.2.6), which arises in the analysis of the wave equation. In this example, the characteristics may be obtained by directly integrating (12.6.3) without using (12.6.2). Each characteristic has the same constant velocity, c_0. The family of characteristics are parallel straight lines, as sketched in Fig. 12.2.1.

Quasi-linear in two-dimensional space. If the independent variables are x and y instead of x and t, then a quasi-linear first-order partial differential equation is usually written in the form

$$a\frac{\partial\rho}{\partial x} + b\frac{\partial\rho}{\partial y} = c, \tag{12.6.4}$$

where a, b, c may be functions of x, y, ρ. The method of characteristics is

$$\frac{d\rho}{dx} = \frac{c}{a}, \tag{12.6.5}$$

if

$$\frac{dy}{dx} = \frac{b}{a}. \tag{12.6.6}$$

This is written in the following (easy to memorize) equivalent form

$$\frac{dx}{a} = \frac{dy}{b} = \frac{d\rho}{c}. \tag{12.6.7}$$

12.6.2 Traffic Flow

Traffic density and flow. As an approximation it is possible to model a congested one-directional highway by a quasi-linear partial differential equation. We introduce the **traffic density** $\rho(x,t)$, the number of cars per mile at time t located at position x. An easily observed and measured quantity is the **traffic flow** $q(x,t)$, the number of cars per hour passing a fixed place x (at time t).

Conservation of cars. We consider an arbitrary section of roadway, between $x = a$ and $x = b$. If there are neither entrances nor exits on this segment of the road, then the number of cars between $x = a$ and $x = b$ ($N = \int_a^b \rho(x,t)\,dx$, the definite integral of the density) might still change in time. The rate of change of the number of cars, dN/dt, equals the number per unit time entering at $x = a$ [the traffic flow $q(a,t)$ there] minus the number of cars per unit time leaving at $x = b$ [the traffic flow $q(b,t)$ there]:

$$\boxed{\frac{d}{dt}\int_a^b \rho(x,t)\,dx = q(a,t) - q(b,t).} \tag{12.6.8}$$

Equation (12.6.8) is called the integral form of **conservation of cars**. As with heat flow, a partial differential equation may be derived from (12.6.8) in several equivalent ways. One way is to note that the boundary contribution may be expressed as an integral over the region:

$$q(a,t) - q(b,t) = -\int_a^b \frac{\partial}{\partial x} q(x,t)\,dx. \tag{12.6.9}$$

Thus, by taking the time-derivative inside the integral (making it a partial derivative) and using (12.6.9) it follows that

$$\boxed{\frac{\partial\rho}{\partial t} + \frac{\partial q}{\partial x} = 0,} \tag{12.6.10}$$

since a and b are arbitrary (see Section 1.2). We call (12.6.10) **conservation of cars.**

Car velocity. The number of cars per hour passing a place equals the density of cars times the velocity of cars. By introducing $u(x,t)$ as the **car velocity**, we have

$$\boxed{q = \rho u.} \tag{12.6.11}$$

In the mid-1950s, Lighthill and Whitham [1955] and, independently, Richards [1956] made a simplifying assumption, namely, that the car velocity depends only on the density, $u = u(\rho)$, with cars slowing down as the traffic density increases; i.e., $du/dp \le 0$. For further discussion, the interested reader is referred to Whitham [1974] and Haberman [1977]. Under this assumption, the traffic flow is only a function of the traffic density, $q = q(\rho)$. In this case, conservation of cars (12.6.10) becomes

$$\boxed{\frac{\partial\rho}{\partial t} + c(\rho)\frac{\partial\rho}{\partial x} = 0,} \tag{12.6.12}$$

where $c(\rho) = q'(\rho)$, a quasi-linear partial differential equation with $Q = 0$ (see (12.6.1)). Here $c(\rho)$ is considered to be a known function of the unknown solution ρ. In any physical problem in which a density ρ is conserved and the flow q is a function of density, ρ satisfies (12.6.12).

12.6.3 Method of Characteristics ($Q = 0$)

The equations for the characteristics for (12.6.12) are

$$\frac{d\rho}{dt} = 0 \tag{12.6.13}$$

along

$$\frac{dx}{dt} = c(\rho). \tag{12.6.14}$$

The characteristic velocity c is not constant but depends on the density ρ. It is known as the **density wave velocity**. From (12.6.13), it follows that the density ρ remains constant along each as yet undetermined characteristic. The velocity of each characteristic, $c(\rho)$, will be constant, since ρ is constant. Each characteristic is thus a straight line (as in the case in which $c(\rho)$ is a constant c_0). However, different characteristics will move at different constant velocities because they may start with different densities. The characteristics, though each is straight, are not parallel to one another. Consider the characteristic that is initially at the position $x = x_0$, as shown in Fig. 12.6.2. Along the curve $dx/dt = c(\rho)$, $d\rho/dt = 0$ or ρ is constant. Initially ρ equals the value at $x = x_0$ (i.e., at $t = 0$). Thus, along this one characteristic,

$$\rho(x,t) = \rho(x_0, 0) = f(x_0), \tag{12.6.15}$$

which is a known constant. The local wave velocity that determines the characteristic is a constant, $dx/dt = c(f(x_0))$. Consequently, this characteristic is a straight line,

$$x = c(f(x_0))t + x_0, \tag{12.6.16}$$

since $x = x_0$ at $t = 0$. Different values of x_0 yield different straight-line characteristics, perhaps as illustrated in Fig. 12.6.2. Along each characteristic, the traffic density ρ is a constant; see (12.6.15). To determine the density at some later time, the characteristic with parameter x_0 that goes through that space-time point must be obtained from (12.6.16).

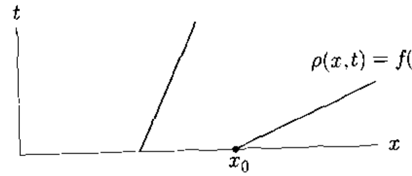

Figure 12.6.2: Possibly nonparallel straight-line characteristics.

Graphical solution. In practice, it is often difficult and not particularly interesting actually to determine x_0 from (12.6.16) as an explicit function of x and t. Instead, a graphical procedure may be used to determine $\rho(x,t)$. Suppose the initial density is as sketched in Fig. 12.6.3. We know that each density ρ_0 stays the same, moving at its own constant density wave velocity $c(\rho_0)$. At time t, the density ρ_0 will have moved a distance $c(\rho_0)t$ as illustrated by the arrow in Fig. 12.6.3. This

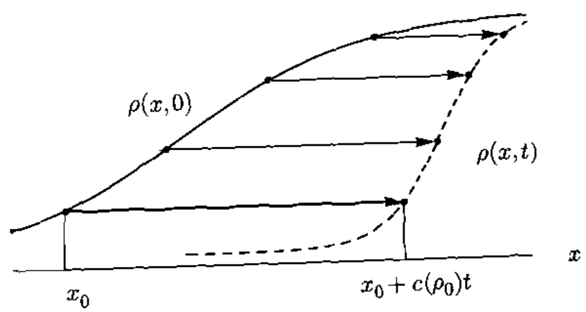

Figure 12.6.3: Graphical solution.

process must be carried out for a large number of points (as is elementary to do on any computer). In this way, we could obtain the density at time t.

Fan-like characteristics. As an example of the method of characteristics, we consider the following initial value problem:

$$\frac{\partial \rho}{\partial t} + 2\rho\frac{\partial \rho}{\partial x} = 0$$

$$\rho(x,0) = \begin{cases} 3 & x < 0 \\ 4 & x > 0. \end{cases}$$

The density $\rho(x,t)$ is constant moving with the characteristic velocity 2ρ:

$$\frac{dx}{dt} = 2\rho.$$

Thus, the characteristics are given by

$$x = 2\rho(x_0, 0)t + x_0. \tag{12.6.17}$$

If $x_0 > 0$, then $\rho(x_0, 0) = 4$, while if $x_0 < 0$ then $\rho(x_0, 0) = 3$. The characteristics, sketched in Fig. 12.6.4, show that

$$\rho(x,t) = \begin{cases} 4 & x > 8t \\ 3 & x < 6t, \end{cases}$$

as illustrated in Fig. 12.6.5. The distance between $\rho = 3$ and $\rho = 4$ is increasing; we refer to the solution as an **expansion wave**. But, what happens for $6t < x < 8t$? The difficulty is caused by the initial density having a discontinuity at $x = 0$. We imagine that all values of ρ between 3 and 4 are present initially at $x = 0$. There will be a straight line characteristic along which ρ equals each value between 3 and 4. Since these characteristics start from $x = 0$ at $t = 0$, it follows from (12.6.17) that the equation for these characteristics is

$$x = 2\rho t, \qquad \text{for } 3 < \rho < 4,$$

also sketched in Fig. 12.6.4. In this way, we obtain the density in the wedge-shaped region

$$\rho = \frac{x}{2t} \qquad \text{for } 6t < x < 8t,$$

which is linear in x (for fixed t). We note that the characteristics fan out from $x = 6t$ to $x = 8t$ and hence are called **fan-like characteristics**. The resulting density is sketched in Fig. 12.6.5. It could also be obtained by the graphical procedure.

12.6.4 Shock Waves

Intersecting characteristics. The method of characteristics will not always work as we have previously described. For quasi-linear partial differential equations, it is quite usual for characteristics to intersect. The resolution will require the

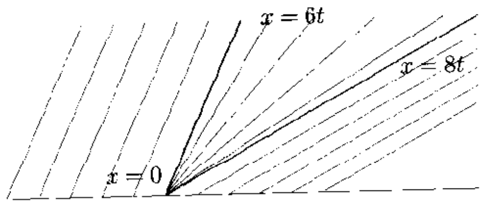

Figure 12.6.4: Characteristics (including the fan-like ones).

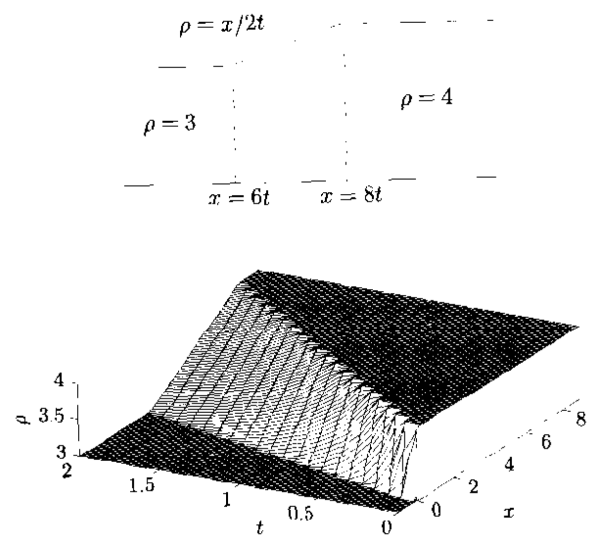

Figure 12.6.5: Expansion wave.

introduction of moving discontinuities called *shock waves*. In order to make the mathematical presentation relatively simple, we restrict our attention to quasi-linear partial differential equations with $Q = 0$, in which case

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \tag{12.6.18}$$

In Fig. 12.6.6 two characteristics are sketched, one starting at $x = x_1$, with $\rho = f(x_1, 0) \equiv \rho_1$ and the other starting at $x = x_2$ with $\rho = f(x_2, 0) \equiv \rho_2$. These characteristics intersect if $c(\rho_1) > c(\rho_2)$, the faster catching up to the slower. The density is constant along characteristics. As time increases, the distance between the densities ρ_1 and ρ_2 decreases. Thus, this is called a **compression wave**. We sketch the initial condition in Fig. 12.6.7(a). The density distribution becomes steeper as time increases [Fig. 12.6.7(b) and (c)]. Eventually characteristics intersect; the theory predicts the density is simultaneously ρ_1 and ρ_2. If we continue to apply

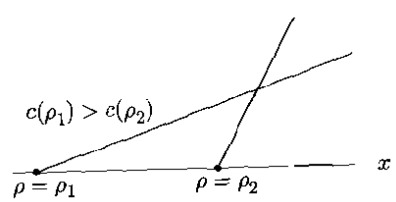

Figure 12.6.6: Intersecting characteristics.

the method of characteristics, the faster-moving characteristic passes the slower. Then we obtain Fig. 12.6.7(d). The method of characteristics predicts that the density becomes a "multivalued" function of position; that is, at some later time our mathematics predicts there will be three densities at some positions [as illustrated in Fig. 12.6.7(d)]. We say the density wave *breaks*. However, in many physical problems (such as traffic flow) it makes no sense to have three values of density at one place.[3] The density must be a single-valued function of position.

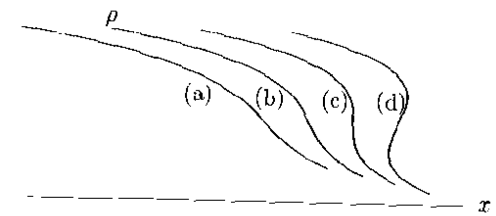

Figure 12.6.7: Density wave steepens (density becomes triple-valued).

Discontinuous solutions. On the basis of the quasi-linear partial differential equation (12.6.18), we predicted the physically impossible phenomenon that the density becomes multivalued. Since the method of characteristics is mathematically justified, it is the partial differential equation itself which must not be entirely valid. Some approximation or assumption that we used must at times be invalid. We will assume that the density (as illustrated in Fig. 12.6.8) and velocity have a jump-discontinuity, which we call a **shock wave**, or simply a **shock**.[4] The shock occurs at some unknown position x_s and propagates in time, so that $x_s(t)$. We introduce the notation x_{s-} and x_{s+} for the position of the shock on the two sides of the discontinuity. The shock velocity, dx_s/dt, is as yet unknown.

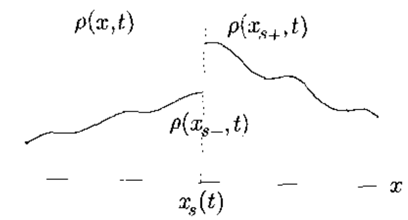

Figure 12.6.8: Density discontinuity at $x = x_s(t)$.

[3]The partial differential equations describing the height of water waves near the shore i.e., in shallow water) are similar to the equations for traffic density waves. In this situation the prediction of breaking is then quite significant!

[4]The terminology *shock wave* is introduced because of the analogous behavior that occurs in gas dynamics. There, changes in pressure and density of air, for example, propagate and are heard (due to the sensitivity of the human ear). They are called *sound waves*. When fluctuations of pressure and density are small, the equations describing sound waves can be linearized. Then sound is propagated at a constant speed known as the *sound speed*. However, if the amplitudes of the fluctuations of pressure and density are not small, then the partial differential equations are quasi-linear. Characteristics may intersect. In this case, the pressure and density can be mathematically modeled as being discontinuous, the result being called a *shock wave*. Examples are the sound emitted from an explosion or the thunder resulting from lightning. If a shock wave results from exceeding the sound barrier, it is known as a *sonic boom*.

Shock velocity. On either side of the shock, the quasi-linear partial differential equation applies, $\partial\rho/\partial t + c(\rho)\partial\rho/\partial x = 0$, where $c(\rho) = dq(\rho)/d\rho$. We need to determine how the discontinuity propagates. If ρ is conserved even at a discontinuity, then the flow relative to the moving shock on one side of the shock must equal the flow relative to the moving shock on the other side. This statement of relative inflow equaling relative outflow becomes

$$\rho(x_{s-},t)\left[u(x_{s-},t)-\frac{dx_s}{dt}\right] = \rho(x_{s+},t)\left[u(x_{s+},t)-\frac{dx_s}{dt}\right], \tag{12.6.19}$$

since flow equals density times velocity (here relative velocity). Solving for the **shock velocity** from (12.6.19) yields

$$\boxed{\frac{dx_s}{dt} = \frac{q(x_{s+},t)-q(x_{s-},t)}{\rho(x_{s+},t)-\rho(x_{s-},t)} = \frac{[q]}{[\rho]},} \tag{12.6.20}$$

where we recall that $q = \rho u$ and where we introduce the notation $[q]$ and $[\rho]$ for the jumps in q and ρ, respectively. In gas dynamics, (12.6.20) is called the Rankine-Hugoniot condition. In summary, **for the conservation law $\partial\rho/\partial t + \partial q/\partial x = 0$ (if the quantity $\int \rho\, dx$ is actually conserved), the shock velocity equals the jump in the flow divided by the jump in the density of the conserved quantity.** At points of discontinuity, this shock condition replaces the use of the partial differential equation, which is valid elsewhere. However, we have not yet explained where shocks occur and how to determine $\rho(x_{s+},t)$ and $\rho(x_{s-},t)$.

Example. We consider the initial value problem

$$\frac{\partial\rho}{\partial t} + 2\rho\frac{\partial\rho}{\partial x} = 0$$

$$\rho(x,0) = \begin{cases} 4 & x < 0 \\ 3 & x > 0. \end{cases}$$

We assume that ρ is a conserved density. Putting the partial differential equation in conservation form $(\partial\rho/\partial t + \partial q/\partial x = 0)$ shows that the flow $q = \rho^2$. Thus, if there is a discontinuity, the shock velocity satisfies $dx/dt = [q]/[\rho] = [\rho^2]/[\rho]$ The density $\rho(x,t)$ is constant moving at the characteristic velocity 2ρ:

$$\frac{dx}{dt} = 2\rho.$$

Therefore, the equation for the characteristics is

$$x = 2\rho(x_0,0)t + x_0.$$

If $x_0 < 0$, then $\rho(x_0,0) = 4$. This parallel group of characteristics intersects those starting from $x_0 > 0$ (with $\rho(x_0,0) = 3$) in the cross-hatched region in

Fig. 12.6.9(a). The method of characteristics yields a multi-valued solution of the partial differential equation. This difficulty is remedied by introducing a shock wave [Fig. 12.6.9(b)], a propagating wave indicating the path at which densities and velocities abruptly change (i.e., are discontinuous). On one side of the shock, the method of characteristics suggests the density is constant $\rho = 4$, and on the other side $\rho = 3$. We do not know as yet the path of the shock. The theory for such a discontinuous solution implies that the path for any shock must satisfy the shock condition, (12.6.20). Substituting the jumps in flow and density yields the following equation for the shock velocity:

$$\frac{dx_s}{dt} = \frac{q(4)-q(3)}{4-3} = \frac{4^2-3^2}{4-3} = 7,$$

since in this case $q = \rho^2$. Thus, the shock moves at a constant velocity. The initial position of the shock is known, giving a condition for this first-order ordinary differential equation. In this case, the shock must initiate at $x_s = 0$ at $t = 0$. Consequently, applying the initial condition results in the position of the shock,

$$x_s = 7t.$$

The resulting space-time diagram is sketched in Fig. 12.6.9(c). For any time $t > 0$, the traffic density is discontinuous, as shown in Fig. 12.6.10.

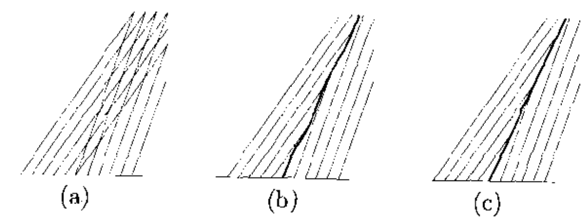

Figure 12.6.9: Shock caused by intersecting characteristics.

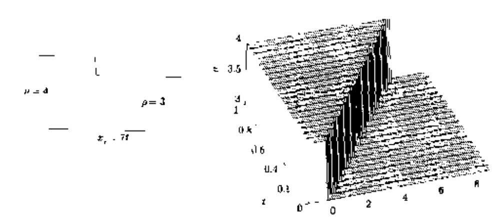

Figure 12.6.10: Density shock wave.

Initiation of a shock. We have described the propagation of shock waves. In the example considered, the density was initially discontinuous; thus, the shock wave formed immediately. However, we will now show that shock waves take a finite time to form if the initial density is continuous. Suppose that the first shock occurs at $t = \tau$ due to the intersection of two characteristics initially a distance Δx (not necessarily small) apart. However, any characteristic starting between the two at $t = 0$ will almost always intersect one of the other two characteristics before $t = \tau$. Thus shocks cannot first occur due to characteristics that are a finite distance Δx apart. Instead, the first shock actually occurs due to the intersection of neighboring characteristics (the limit as $\Delta x \to 0$). We will show that even though $\Delta x \to 0$, the first intersection occurs at a finite positive time, the time of the earliest shock. The density ρ is constant along characteristics, satisfying $dx/dt = c(\rho)$. We will analyze neighboring characteristics. Consider the characteristic emanating from $x = x_0$ at $t = 0$, where $\rho(x, 0) = f(x)$,

$$x = c[f(x_0)]t + x_0 \qquad (12.6.21)$$

and the characteristic starting from $x = x_0 + \Delta x$ at $t = 0$,

$$x = c[f(x_0 + \Delta x)]t + x_0 + \Delta x.$$

Only if $c[f(x_0)] > c[f(x_0 + \Delta x)]$ will these *characteristics intersect* (in a positive time). Solving for the intersection point by eliminating x yields

$$c[f(x_0)]t + x_0 = c[f(x_0 + \Delta x)]t + x_0 + \Delta x.$$

Therefore, the time at which nearly neighboring curves intersect is

$$t = \frac{\Delta x}{c[f(x_0)] - c[f(x_0 + \Delta x)]} = \frac{1}{\{c[f(x_0)] - c[f(x_0 + \Delta x)]\}/\Delta x}.$$

The characteristics are paths of observers following constant density. Then this equation states that the time of intersection of the two characteristics is the initial distance between the characteristics divided by the relative velocity of the two characteristics. Although the distance in between is small, the relative velocity is also small. To consider neighboring characteristics, the limit as $\Delta x \to 0$ must be calculated:

$$\boxed{t = \frac{-1}{dc/dx_0}.} \qquad (12.6.22)$$

Characteristics will intersect ($t > 0$) only if $(d/dx_0)c[f(x_0)] < 0$. Thus, we conclude that all neighboring characteristics that emanate from regions where the characteristic velocity is *locally* decreasing will always intersect. To determine the first time at which an intersection (shock) occurs, we must minimize the intersection time over all possible neighboring characteristics. i.e., find the *absolute* minimum of t given by equation (12.6.22). This can be calculated by determining where $d^2/dx_0^2 c[f(x_0)] = 0$.

Shock dynamics. We will show that the slope of the solution is infinite where neighboring characteristics intersect. Since $\rho(x,t) = \rho(x_0, 0)$, we have

$$\frac{\partial \rho}{\partial x} = \frac{d\rho}{dx_0}\frac{\partial x_0}{\partial x} = \frac{d\rho}{dx_0} \Big/ \left[1 + \frac{dc}{dx_0}t\right].$$

This has also used the result of partial differentiation of (12.6.21) with respect to x:

$$1 = \frac{\partial x_0}{\partial x}\left[1 + \frac{dc}{dx_0}t\right].$$

The slope is infinite at those places satisfying (12.6.22). This shows that the turning points of the triple-valued solution correspond to the intersection of neighboring characteristics (the envelope of the characteristics). Within the envelope of characteristics, the solution is triple-valued. It is known that the envelope of the characteristics is cusp-shaped, as indicated in Fig. 12.6.11. However, the triple-valued solution (within the cusp region) makes no sense. Instead, as discussed earlier, a shock wave exists satisfying (12.6.20), initiating at the cusp point. The shock is located within the envelope. In fact, the triple-valued solution, obtained by the method of characteristics, may be used to determine the location of the shock. Whitham [1974] has shown that the correct location of the shock may be determined by cutting the lobes off to form equal areas (Fig. 12.6.12). The reason for this is that the method of characteristics conserves cars and that, when a shock is introduced, the number of cars (represented by the area $\int \rho\, dx$) must also be the same as it is initially.

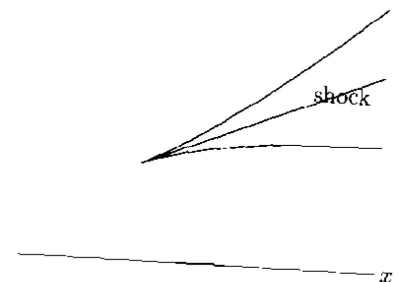

Figure 12.6.11: Envelope of characteristics, locus of intersections of neighboring characteristics.

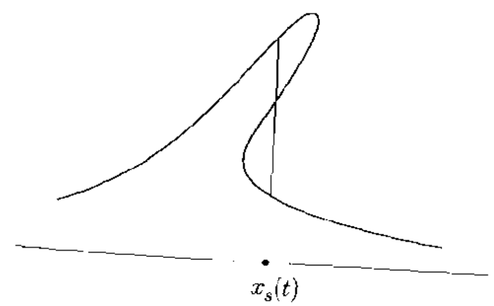

Figure 12.6.12: Whitham's equal-area principle.

12.6.5 Quasi-Linear Example

Consider the quasi-linear example

$$\frac{\partial \rho}{\partial t} - \rho\frac{\partial \rho}{\partial x} = -2\rho, \qquad (12.6.23)$$

subject to the initial conditions

$$\rho(x,0) = f(x). \tag{12.6.24}$$

This could model a traffic flow problem (with ρ a scaled version of the density) where cars are not conserved but instead leave the roadway (at exits) at a rate proportional to the density (as though cars exit the highway to avoid congestion).

The method of characteristics yields

$$\frac{d\rho}{dt} = -2\rho, \tag{12.6.25}$$

along the characteristic

$$\frac{dx}{dt} = -\rho. \tag{12.6.26}$$

These equations are sometimes written in the equivalent form

$$\frac{d\rho}{-2\rho} = \frac{dx}{-\rho} = dt. \tag{12.6.27}$$

Sometimes the coupled system of ordinary differential equations can be directly solved. In this case the ordinary differential equation for ρ may be solved first, and its solution used to determine the characteristic x. Because of the initial condition (12.6.24), we introduce the parameter x_0 representing the characteristic that emanates from $x = x_0$ (at $t = 0$). From (12.6.25), along the characteristic, we obtain

$$\rho(x,t) = \rho(x_0,0)e^{-2t} = f(x_0)e^{-2t}. \tag{12.6.28}$$

The parameter x_0 is constant along each characteristic. The solution (density) exponentially decays along the characteristic as time increases. Thus, the characteristic velocity becomes

$$\frac{dx}{dt} = -f(x_0)e^{-2t}.$$

By integrating the velocity, we obtain the position of the characteristic:

$$x = \frac{1}{2}f(x_0)e^{-2t} - \frac{1}{2}f(x_0) + x_0, \tag{12.6.29}$$

since x must equal x_0 at $t = 0$. Here, the characteristics are not straight lines. The parametric representation of the solution is obtained from (12.6.28), where x_0 should be considered as a function of x and t from (12.6.29). Usually an explicit solution is impractical.

Explicit solution of an initial value problem. For the quasi-linear partial differential equation (12.6.23), suppose the initial conditions are

$$\rho(x,0) = f(x) = x.$$

In this case, from (12.6.29), the characteristics satisfy

$$x = \frac{1}{2}x_0 e^{-2t} + \frac{1}{2}x_0.$$

Thus, an explicit solution can be obtained

$$x_0 = \frac{2x}{1+e^{-2t}}.$$

Note that for each x and t there is only one characteristic x_0 (because the initial condition was chosen such that the family of characteristics does not intersect itself.) From (12.6.28), the solution of the initial value problem for the partial differential equation is

$$\rho(x,t) = \frac{2xe^{-2t}}{1+e^{-2t}} = \frac{2x}{1+e^{2t}}.$$

General solution. Quasi-linear partial differential equations have general solutions (as with the linear wave equation). Without specifying initial conditions, integrating (12.6.25) and (12.6.26) yields

$$\begin{aligned} \rho(x,t) &= c_1 e^{-2t} \\ x &= c_1 e^{-2t} + c_2. \end{aligned}$$

In general, one constant can be an arbitrary function of the other contant. Thus, we obtain the **general solution** of (12.6.23),

$$x = \rho + f(\rho e^{2t}).$$

EXERCISES 12.6

12.6.1. Determine the solution $\rho(x,t)$ satisfying the initial condition $\rho(x,0) = f(x)$ if

- *(a) $\frac{\partial\rho}{\partial t} = 0$
- (b) $\frac{\partial\rho}{\partial t} = -3\rho + 4e^{7t}$
- *(c) $\frac{\partial\rho}{\partial t} = -3x\rho$
- (d) $\frac{\partial\rho}{\partial t} = x^2 t\rho$

***12.6.2.** Determine the solution of $\partial\rho/\partial t = \rho$, which satisfies $\rho(x,t) = 1 + \sin x$ along $x = -2t$.

12.6.3. Suppose $\frac{\partial\rho}{\partial t} + c_0\frac{\partial\rho}{\partial x} = 0$ with c_0 constant.

- *(a) Determine $\rho(x,t)$ if $\rho(x,0) = \sin x$.
- *(b) If $c_0 > 0$, determine $\rho(x,t)$ for $x > 0$ and $t > 0$, where $\rho(x,0) = f(x)$ for $x > 0$ and $\rho(0,t) = g(t)$ for $t > 0$.
- (c) Show that part (b) cannot be solved if $c_0 < 0$.

***12.6.4.** If $u(\rho) = \alpha + \beta\rho$, determine α and β such that $u(0) = u_{\max}$ and $u(\rho_{\max}) = 0$.

- (a) What is the flow as a function of density? Graph the flow as a function of the density.

(b) At what density is the flow maximum? What is the corresponding velocity? What is the maximum flow (called the *capacity*)?

12.6.5. Redo Exercise 12.6.4 if $u(\rho) = u_{\max}(1 - \rho^3/\rho_{\max}^3)$.

12.6.6. Consider the traffic flow problem

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0.$$

Assume $u(\rho) = u_{\max}(1 - \rho/\rho_{\max})$. Solve for $\rho(x,t)$ if the initial conditions are

(a) $\rho(x,0) = \rho_{\max}$ for $x < 0$ and $\rho(x,0) = 0$ for $x > 0$. This corresponds to the traffic density that results after an infinite line of stopped traffic is started by a red light turning green.

(b) $\rho(x,0) = \begin{cases} \rho_{\max} & x < 0 \\ \frac{\rho_{\max}}{2} & 0 < x < a \\ 0 & x > a \end{cases}$

(c) $\rho(x,0) = \begin{cases} \frac{3\rho_{\max}}{5} & x < 0 \\ \frac{\rho_{\max}}{5} & x > 0 \end{cases}$

12.6.7. Solve the following problems:

(a) $\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial \rho}{\partial x} = 0 \qquad \rho(x,0) = \begin{cases} 3 & x < 0 \\ 4 & x > 0 \end{cases}$

(b) $\frac{\partial \rho}{\partial t} + 4\rho \frac{\partial \rho}{\partial x} = 0 \qquad \rho(x,0) = \begin{cases} 2 & x < 1 \\ 3 & x > 1 \end{cases}$

(c) $\frac{\partial \rho}{\partial t} + 3\rho \frac{\partial \rho}{\partial x} = 0 \qquad \rho(x,0) = \begin{cases} 1 & x < 0 \\ 2 & 0 < x < 1 \\ 4 & x > 1 \end{cases}$

(d) $\frac{\partial \rho}{\partial t} + 6\rho \frac{\partial \rho}{\partial x} = 0$ for $x > 0$ only $\qquad \begin{matrix} \rho(x,0) = 5 & x > 0 \\ \rho(0,t) = 2 & t > 0 \end{matrix}$

12.6.8. Solve subject to the initial condition $\rho(x,0) = f(x)$

*(a) $\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = e^{-3x}$

(b) $\frac{\partial \rho}{\partial t} + 3x\frac{\partial \rho}{\partial x} = 4$

*(c) $\frac{\partial \rho}{\partial t} + t\frac{\partial \rho}{\partial x} = 5$

(d) $\frac{\partial \rho}{\partial t} + 5t\frac{\partial \rho}{\partial x} = 3\rho$

*(e) $\frac{\partial \rho}{\partial t} - t^2\frac{\partial \rho}{\partial x} = -\rho$

(f) $\frac{\partial \rho}{\partial t} + t^2\frac{\partial \rho}{\partial x} = 0$

*(g) $\frac{\partial \rho}{\partial t} + x\frac{\partial \rho}{\partial x} = t$

12.6.9. Determine a parametric representation of the solution satisfying $\rho(x,0) = f(x)$:

*(a) $\frac{\partial \rho}{\partial t} - \rho^2\frac{\partial \rho}{\partial x} = 3\rho$

(b) $\frac{\partial \rho}{\partial t} + \rho\frac{\partial \rho}{\partial x} = t$

*(c) $\frac{\partial \rho}{\partial t} + t^2\rho\frac{\partial \rho}{\partial x} = -\rho$

(d) $\frac{\partial \rho}{\partial t} + \rho\frac{\partial \rho}{\partial x} = -x\rho$

12.6.10. Solve $\frac{\partial \rho}{\partial t} + t^2\frac{\partial \rho}{\partial x} = 4\rho$ for $x > 0$ and $t > 0$ with $\rho(0,t) = h(t)$ and $\rho(x,0) = 0$.

12.6.11. Solve $\frac{\partial \rho}{\partial t} + (1+t)\frac{\partial \rho}{\partial x} = 3\rho$ for $t > 0$ and $x > -t/2$ with $\rho(x,0) = f(t)$ for $x > 0$ and $\rho(x,t) = g(t)$ along $x = -t/2$.

12.6.12. Consider (12.6.8) if there is a moving shock x_s such that $a < x_s(t) < b$. By differentiating the integral [with a discontinuous integrand at $x_s(t)$], *derive* (12.6.20).

12.6.13. Suppose that, instead of $u = U(\rho)$, a car's velocity u is

$$u = U(\rho) - \frac{\nu}{\rho}\frac{\partial \rho}{\partial x},$$

where ν is a constant.

(a) What sign should ν have for this expression to be physically reasonable?

(b) What equation now describes conservation of cars?

(c) Assume that $U(\rho) = u_{\max}(1 - \rho/\rho_{\max})$. Derive **Burgers' equation**:

$$\frac{\partial \rho}{\partial t} + u_{\max}\left[1 - \frac{2\rho}{\rho_{\max}}\right]\frac{\partial \rho}{\partial x} = \nu\frac{\partial^2 \rho}{\partial x^2}. \qquad (12.6.30)$$

12.6.14. Consider Burgers' equation as derived in Exercise 12.6.13. Suppose that a solution exists as a density wave moving without change of shape at velocity V, $\rho(x,t) = f(x - Vt)$.

*(a) What ordinary differential equation is satisfied by f?

(b) Integrate this differential equation once. By graphical techniques show that a solution exists such that $f \to \rho_2$ as $x \to +\infty$ and $f \to \rho_1$ as $x \to -\infty$ only if $\rho_2 > \rho_1$. Roughly sketch this solution. Give a physical interpretation of this result.

*(c) Show that the velocity of wave propagation, V, is the same as the shock velocity separating $\rho = \rho_1$ from $\rho = \rho_2$ (occurring if $\nu = 0$).

12.6.15. Consider Burgers' equation as derived in Exercise 12.6.13. Show that the change of dependent variables

$$\rho = \frac{\nu\rho_{\max}}{u_{\max}}\frac{\phi_x}{\phi},$$

introduced independently by E. Hopf and J. D. Cole, transforms Burgers' equation into a diffusion equation, $\frac{\partial \phi}{\partial t} + u_{\max}\frac{\partial \phi}{\partial x} = \nu\frac{\partial^2 \phi}{\partial x^2}$. Use this to solve the initial value problem $\rho(x,0) = f(x)$ for $-\infty < x < \infty$. [In Whitham [1974] it is shown that this exact solution can be asymptotically analyzed as $\nu \to 0$ using Laplace's method for exponential integrals to show that $\rho(x,t)$ approaches the solution obtained for $\nu = 0$ using the method of characteristics with shock dynamics.]

12.6.16. Suppose that the initial traffic density is $\rho(x,0) = \rho_0$ for $x < 0$ and $\rho(x,0) = \rho_1$ for $x > 0$. Consider the two cases, $\rho_0 < \rho_1$ and $\rho_1 < \rho_0$. For which of the preceding cases is a density shock necessary? Briefly explain.

12.6.17. Consider a traffic problem, with $u(\rho) = u_{\max}(1 - \rho/\rho_{\max})$. Determine $\rho(x,t)$ if

$$*(a)\quad \rho(x,0) = \begin{cases} \frac{\rho_{\max}}{5} & x < 0 \\ \frac{3\rho_{\max}}{5} & x > 0 \end{cases} \qquad (b)\quad \rho(x,0) = \begin{cases} \frac{\rho_{\max}}{3} & x < 0 \\ \frac{2\rho_{\max}}{3} & x > 0 \end{cases}$$

12.6.18. Assume that $u(\rho) = u_{\max}(1 - \rho^2/\rho_{\max}^2)$. Determine the traffic density ρ (for $t > 0$) if $\rho(x,0) = \rho_1$ for $x < 0$ and $\rho(x,0) = \rho_2$ for $x > 0$.

(a) Assume that $\rho_2 > \rho_1$. *(b) Assume that $\rho_2 < \rho_1$.

12.6.19. Solve the following problems:

(a) $\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial \rho}{\partial x} = 0$ $\qquad \rho(x,0) = \begin{cases} 4 & x < 0 \\ 3 & x > 0 \end{cases}$

(b) $\frac{\partial \rho}{\partial t} + 4\rho \frac{\partial \rho}{\partial x} = 0$ $\qquad \rho(x,0) = \begin{cases} 3 & x < 1 \\ 2 & x > 1 \end{cases}$

(c) $\frac{\partial \rho}{\partial t} + 3\rho \frac{\partial \rho}{\partial x} = 0$ $\qquad \rho(x,0) = \begin{cases} 4 & x < 0 \\ 2 & 0 < x < 1 \\ 1 & x > 1 \end{cases}$

(d) $\frac{\partial \rho}{\partial t} + 6\rho \frac{\partial \rho}{\partial x} = 0$ for $x > 0$ only $\qquad \begin{matrix} \rho(x,0) = 2 & x > 0 \\ \rho(0,t) = 5 & t > 0 \end{matrix}$

12.7 First-Order Nonlinear Partial Differential Equations

12.7.1 Derive Eikonal Equation from Wave Equation

For simplicity we consider the two - dimensional wave equation

$$\frac{\partial^2 E}{\partial t^2} = c^2\left(\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2}\right). \tag{12.7.1}$$

Plane waves and their reflections were analyzed in Section 4.6. Nearly plane waves exist under many circumstances. If the coefficient c is not constant but varies slowly, then over a few wave lengths the wave sees nearly constant c. However, over long distances (relative to short wave lengths) we may be interested in the effects of variable c. Another situation in which nearly plane waves arise is the reflection of a plane wave by a curved boundary (or reflection and refraction by a curved interface between two medias with different indices of refraction). We assume the radius of curvature of the boundary is much longer than typical wave lengths. In many of these situations the temporal frequency ω is fixed (by an incoming plane wave). Thus,

$$E = A(x,y)e^{-i\omega t}, \tag{12.7.2}$$

where $A(x,y)$ satisfies the **Helmholtz** or **reduced wave equation**:

$$-\omega^2 A = c^2\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}\right). \tag{12.7.3}$$

Again the temporal frequency ω is fixed (and given) but $c = c(x,y)$ for inhomogeneous media or $c =$ constant for uniform media.

In uniform media ($c =$ constant), plane waves of the form $E = A_0 e^{i(k_1 x + k_2 y - \omega t)}$ or

$$A = A_0 e^{i(k_1 x + k_2 y)} \tag{12.7.4}$$

exist if

$$\omega^2 = c^2(k_1^2 + k_2^2). \tag{12.7.5}$$

For nearly plane waves, we introduce the phase $u(x,y)$ of the reduced wave equation:

$$A(x,y) = R(x,y)e^{iu(x,y)}. \tag{12.7.6}$$

The wave numbers k_1 and k_2 for uniform media are usually called p and q respectively and are defined by

$$p = \frac{\partial u}{\partial x} \tag{12.7.7}$$

$$q = \frac{\partial u}{\partial y}. \tag{12.7.8}$$

As an approximation (which can be derived using perturbation methods), it can be shown that the (slowly varying) wave numbers satisfy (12.7.5), corresponding to the given temporal frequency associated with plane waves,

$$\boxed{\omega^2 = c^2(p^2 + q^2).} \tag{12.7.9}$$

This is a first order nonlinear partial differential equation (not quasi-linear) for the phase $u(x,y)$, known as the **eikonal equation**

$$\boxed{\frac{\omega^2}{c^2} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2,} \tag{12.7.10}$$

where ω is a fixed reference temporal frequency and $c = c(x, y)$ for inhomogeneous media or $c =$ constant for uniform media. Sometimes the **index of refraction** $n(x, y)$ is introduced proportional to $\frac{1}{c}$. The amplitude $R(x, y)$ solves equations (which we do not discuss) known as the transport equations, which describe the propagation of energy of these nearly plane waves.

12.7.2 Solving the Eikonal Equation in Uniform Media and Reflected Waves

The simplest example of the eikonal equation (12.7.10) occurs in uniform media ($c =$ constant):

$$\boxed{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \frac{\omega^2}{c^2},} \tag{12.7.11}$$

where ω and c are constants. Rather than solve for $u(x, y)$ directly, we will show that it is easier to solve first for $p = \frac{\partial u}{\partial x}$ and $q = \frac{\partial u}{\partial y}$. Thus, we consider

$$p^2 + q^2 = \frac{\omega^2}{c^2}. \tag{12.7.12}$$

Differentiating (12.7.11) or (12.7.12) with respect to x yields

$$p\frac{\partial p}{\partial x} + q\frac{\partial q}{\partial x} = 0.$$

Since $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$, p satisfies a first-order quasi-linear partial differential equation

$$p\frac{\partial p}{\partial x} + q\frac{\partial p}{\partial y} = 0. \tag{12.7.13}$$

Equation (12.7.13) may be solved by the method of characteristics [see (12.6.7)]

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dp}{0}. \tag{12.7.14}$$

If there is a boundary condition for p, then (12.7.14) can be solved for p since $q = \pm\sqrt{\frac{\omega^2}{c^2} - p^2}$ (from (12.7.12). Since (12.7.14) shows that p is constant along each characteristic, it also follows from (12.7.14) that each characteristic is a straight line. In this way p can be determined. However, given $p = \frac{\partial u}{\partial x}$, integrating for u is not completely straightforward.

We have differentiated the eikonal equation with respect to x. If instead we differentiate with respect to y we obtain:

$$p\frac{\partial p}{\partial y} + q\frac{\partial q}{\partial y} = 0.$$

A first-order quasi-linear partial differential equation for q can be obtained by again using $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$:

$$p\frac{\partial q}{\partial x} + q\frac{\partial q}{\partial y} = 0. \tag{12.7.15}$$

Thus, $\frac{dx}{p} = \frac{dy}{q} = \frac{dq}{0}$, which when combined with (12.7.14) yields the more general result

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dp}{0} = \frac{dq}{0}. \tag{12.7.16}$$

However, usually we want to determine u so that we wish to determine how u varies along this characteristic: $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = p\,dx + q\,dy = p^2\frac{dx}{p} + q^2\frac{dy}{p} = (p^2 + q^2)\frac{dx}{p} = \frac{\omega^2}{c^2}\frac{dx}{p}$, where we have used (12.7.16) and (12.7.12). Thus, for the eikonal equation

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dp}{0} = \frac{dq}{0} = \frac{du}{\omega^2/c^2}. \tag{12.7.17}$$

The characteristics are straight lines since p and q are constants along the characteristics.

Reflected waves. We consider an elementary incoming plane wave $e^{i(\boldsymbol{k}_I \cdot \boldsymbol{x} - \omega t)}$ where $\boldsymbol{k}_I$ represents the given constant incoming wave number vector and where $\omega = c|\boldsymbol{k}_I|$. We assume the plane wave reflects off a curved boundary (as illustrated in Figure 12.7.1) which we represent with a parameter τ as $x = x_0(\tau)$ and $y = y_0(\tau)$. We introduce the unknown reflected wave, $R(x,t)e^{iu(x,y)}e^{-i\omega t}$, and we wish to determine the phase $u(x, y)$ of the reflected wave. The eikonal equation

$$p^2 + q^2 = \frac{\omega^2}{c^2} = |\boldsymbol{k}_I|^2$$

can be interpreted as saying the slowly varying reflected wave number vector (p, q) has the same length as the constant incoming wave number vector (physically the slowly varying reflected wave will always have the same wave length as the incident wave). We assume the boundary condition on the curved boundary is that the total field is zero (other boundary conditions yield the same equations for the phase): $0 = e^{i(\boldsymbol{k}_I \cdot \boldsymbol{x} - \omega t)} + R(x,t)e^{iu(x,y)}e^{-i\omega t}$. Thus, on the boundary the phase of the incoming wave and the phase of the reflected wave must be the same:

$$u(x_0, y_0) = \boldsymbol{k}_I \cdot \boldsymbol{x}_0. \tag{12.7.18}$$

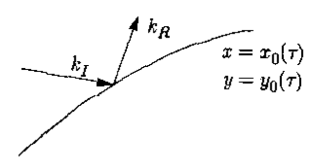

Figure 12.7.1: Reflected wave from curved boundary.

Taking the derivative of (12.7.18) with respect to the parameter τ shows that

$$\frac{\partial u}{\partial x}\frac{dx_0}{d\tau}+\frac{\partial u}{\partial y}\frac{dy_0}{d\tau} = p\frac{dx_0}{d\tau}+q\frac{dy_0}{d\tau} = \boldsymbol{k}_R\cdot\frac{d\boldsymbol{x}_0}{d\tau} = \boldsymbol{k}_I\cdot\frac{d\boldsymbol{x}_0}{d\tau}, \tag{12.7.19}$$

where we have noted that the vector (p, q) is the unknown reflected wave number vector $\boldsymbol{k}_R$ (because p and q are constant along the characteristic). Since $d\boldsymbol{x}_0/d\tau$ is a vector tangent to the boundary, (12.7.19) shows that the tangential component of the incoming and reflecting wave numbers must be the same. Since the magnitude of the incident and reflecting wave number vectors are the same, it follows that the normal component of the reflected wave must be minus the normal component of the incident wave. Thus, the angle of reflection off a curved boundary is the same as the angle of incidence. Thus at any point along the boundary the constant value of p and q is known for the reflected wave. Because $q = \pm\sqrt{\frac{\omega^2}{c^2} - p^2}$ there are two solutions of the eikonal equation: one represents the incoming wave and the other (of interest to us) the reflected wave.

To obtain the phase of the reflected wave, we must solve the characteristic equations (12.7.17) for the eikonal equation with the boundary condition specified by (12.7.18). Since for uniform media $\frac{\omega^2}{c^2} = |\boldsymbol{k}_I|^2$ is a constant, the differential equation for u along the characteristic, $\frac{du}{d\tau} = \frac{\omega^2}{c^2}\frac{1}{p} = \frac{|\boldsymbol{k}_I|^2}{p}$, can be integrated (since p is constant) using the boundary condition to give

$$u(x,y) = \frac{|\boldsymbol{k}_I|^2}{p}(x - x_0) + \boldsymbol{k}_I\cdot\boldsymbol{x}_0,$$

along a specific characteristic. The equation for the characteristics $p(y - y_0) = q(x - x_0)$ corresponds to the angle of reflection equaling the angle of incidence. Since $p^2 + q^2 = |\boldsymbol{k}_I|^2$, the more pleasing representation of the phase (solution of the eikonal equation) follows along a specific characteristic:

$$u(x,y) = p(x - x_0) + q(y - y_0) + \boldsymbol{k}_I\cdot\boldsymbol{x}_0, \tag{12.7.20}$$

where $u(x_0, y_0) = \boldsymbol{k}_I\cdot\boldsymbol{x}_0$ is the phase of the incident wave on the boundary.

12.7.3 First-Order Nonlinear Partial Differential Equations

Any first-order nonlinear partial differential equation can be put in the form

$$\boxed{F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0.} \tag{12.7.21}$$

As with the eikonal equation example of the previous subsection, we show tha $p = \frac{\partial u}{\partial x}$ and $q = \frac{\partial u}{\partial y}$ solve quasi-linear partial differential equations, and henc (12.7.21) can be solved by the method of characteristics. Using p and q gives

$$F(x, y, u, p, q) = 0. \quad (12.7.2$$

Taking the partial derivative of (12.7.22) with respect to x, we obtain

$$F_x + F_u p + F_p\frac{\partial p}{\partial x} + F_q\frac{\partial q}{\partial x} = 0,$$

where we use the subscript notation for partial derivatives. For example $F_u \equiv \frac{\partial F}{\partial u}$ keeping x, y, p, q constant. Since $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$, we obtain a quasi-linear partial differential equation for p:

$$F_p\frac{\partial p}{\partial x} + F_q\frac{\partial p}{\partial y} = -F_x - F_u p.$$

Thus, the method of characteristics for p yields

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dp}{-F_x - F_u p}. \tag{12.7.23}$$

Similarly, taking the partial derivative of (12.7.22) with respect to y, yields

$$F_y + F_u q + F_p\frac{\partial p}{\partial y} + F_q\frac{\partial q}{\partial y} = 0.$$

Here $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$ yields a quasi-linear partial differential equation for q:

$$F_p\frac{\partial q}{\partial x} + F_q\frac{\partial q}{\partial y} = -F_y - F_u q.$$

The characteristic direction is the same as in (12.7.23), so that (12.7.23) is amended to become

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dp}{-F_x - F_u p} = \frac{dq}{-F_y - F_u q}. \tag{12.7.24}$$

In order to solve for u, we want to derive a differential equation for $u(x, y)$ along the characteristics:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = p\,dx + q\,dy = pF_p\frac{dx}{F_p} + qF_q\frac{dy}{F_q} = (pF_p + qF_q)\frac{dx}{F_p}.$$

The complete system to solve for p, q, and u is

$$\boxed{\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dp}{-F_x - F_u p} = \frac{dq}{-F_y - F_u q} = \frac{du}{pF_p + qF_q}.} \tag{12.7.25}$$