The Method of Characteristics for first order q-l. PDEs

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$u(x,y)$ unknown function

$A(x,y,u)$, $B(x,y,u)$, $C(x,y,u)$,

coefficients - given functions

$$A(x,y,u)\,u_x(x,y) + B(x,y,u)\,u_y(x,y) = C(x,y,u)$$

$$\boxed{Au_x + Bu_y = C}$$

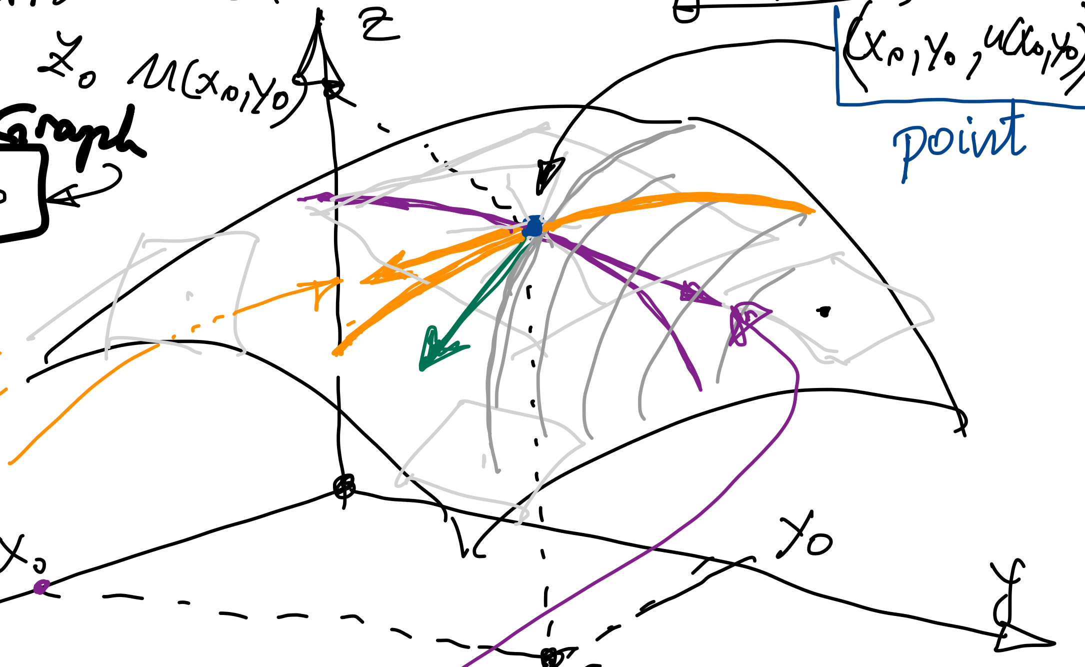
First order quasilinear PDE

Let us establish a connection to a system of three ordinary differential equations, The connection is geometric.

Prereq. first Math 224 functions of two variables and their graphs.

$z = u(x,y)$ Graph

$\boxed{(x, y, u(x,y)) \text{ all points}}$



orange curve

$(x, y_0, u(x, y_0))$

tangent vector

$\langle 1, 0, u_x(x_0, y_0) \rangle$

purple curve:

$(x_0, y, u(x_0, y))$

$\langle 0, 1, u_y(x_0, y_0) \rangle$ tangent vector

At any point $(x, y, u(x,y))$ we have two vectors

$$\langle 1, 0, u_x(x,y) \rangle \quad \text{and} \quad \langle 0, 1, u_y(x,y) \rangle$$

$$\langle 1, 0, u_x \rangle \quad \text{and} \quad \langle 0, 1, u_y \rangle$$

(abbreviated)

I need more space for the dramatic development that follows. The main actors are vectors $\langle A, B, C \rangle$, $\langle 1, 0, u_x \rangle$, $\langle 0, 1, u_y \rangle$

$$\boxed{\langle A, B, C\rangle} = \langle A, B, Au_x + Bu_y\rangle$$

remember our PDE

$C = Au_x + Bu_y$

$$= \langle A, 0, Au_x\rangle + \langle 0, B, Bu_y\rangle$$

remember vector algebra and use it to separate A & B

$$\boxed{= A\langle 1, 0, u_x\rangle + B\langle 0, 1, u_y\rangle}$$

vector algebra? to separate red & green

rewrite

$$\langle A, B, C\rangle = A\langle 1, 0, u_x\rangle + B\langle 0, 1, u_y\rangle$$

$$\vec{a} = \alpha\vec{u} + \beta\vec{v}$$

$\langle A, B, C \rangle$ vector is a lin. comb of orange and purple

(at the begining)
We are given the vector field $\langle A, B, C \rangle$.

In Math 331 you have learned how to find CURVES that are tangent to a given vect. field.

characteristic equations for the quasilinear PDE

$$\begin{cases} X'(s) = A(X, Y, Z) \\ Y'(s) = B(X, Y, Z) \\ Z'(s) = C(X, Y, Z) \end{cases}$$

to a get a specific curve I need a given point

$$\begin{cases} X(0) = ? \\ Y(0) = ? \\ Z(0) = ? \end{cases}$$

Initial condition