$$\frac{\partial u}{\partial t}(x,t) = \kappa \frac{\partial^2 u}{\partial x^2}(x,t)$$

$k$ \kappa

$$\sum_{k=1}^{n} a_k$$

Divchlet

BCs $u(0,t) = 0 \quad u(L,t) = 0 \quad \forall t \geq 0$

IC $u(x,0) = f(x) \quad 0 \leq x \leq L$

$$u(x,t) = \underbrace{A(x)\,B(t)}$$

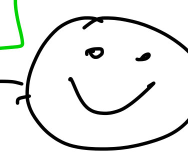
"high" level of sophistication

two objects of lower level of sophistication

$$A(0)B(t) = 0, \quad A(L)B(t) = 0 \quad \forall t$$

$$\boxed{A(0) = 0 \qquad A(L) = 0}$$

I have info about A



$$\boxed{A(x)\,B(0) = f(x)}$$

Now PDE $A(x)B'(t) = k\,A''(x)B(t)$

Drama:

$$\frac{B'(t)}{k\,B(t)} = \frac{A''(x)}{A(x)} = -\lambda$$

no change

only $t$ only $x$

constant which I call $-\lambda$

$B'(t) = -k\lambda B(t)$

$$\boxed{B(t) = C_0 e^{-k\lambda t}}$$

DONE

$A''(x) = -\lambda A(x)$

$$\boxed{\begin{gathered} -A''(x) = \lambda A(x) \\ A(0) = 0,\ A(L) = 0 \end{gathered}}$$

This is our problem now

This looks exactly like a matrix eigenvalue problem in Linear Algebra

$M\vec{v} = \lambda\vec{v}$ $\vec{v} \neq \vec{0}$

What is $M$ in our problem?

$\boxed{-\frac{d^2}{dx^2}}$ is our $M$

+ BCs at 0 and $L$

Below we find eigenvalues: $\lambda_m = \left(\frac{m\pi}{L}\right)^2$ $m \in \mathbb{N}$

and eigenfunctions $\sin\left(\frac{m\pi}{L}x\right)$

---

finding eigenfunctions and eigenvalues for

$-\frac{d^2}{dx^2}$ + Dirichlet boundary conditions at 0 and $L$

Solve $A''(x) + \lambda A(x) = 0$ subject to $A(0) = A(L) = 0$

We use BK (background knowledge) solving $y'' + \beta y = 0$ 2nd order ode.

Case 1 $\lambda > 0$. My style set $\boxed{\lambda = \mu^2, \ \mu \geq 0}$

$A''(x) + \mu^2 A(x) = 0$. The fundamental set of solutions is $\cos(\mu x)$, $\sin(\mu x)$. Thus the general sol. is ALL

$$\boxed{A(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x)}$$

$\mu \geq 0$

Satisfy BCs: $0 = A(0) = C_1 \Rightarrow C_1 = 0$

$0 = A(L) = C_2 \sin(\mu L) \Rightarrow \mu L = m\pi$, $m \in \mathbb{N}$

$$\boxed{\mu = \frac{m\pi}{L}, \ m \in \mathbb{N}}$$

Now I get an equation for $\lambda$.

eigenvalues

$$\boxed{\lambda_m = \left(\frac{m\pi}{L}\right)^2, \ m \in \mathbb{N}}$$

We obtain a lot of $\lambda$-s

the corresponding $A$-s are $\boxed{A_m(x) = \sin\left(\frac{m\pi}{L}x\right), \ m \in \mathbb{N}}$

<u>Case 2</u> $\lambda = 0$ Is $\lambda=0$ an eigenvalue.

$A'' + 0 \cdot A = 0$, has the fundamental set of solutions $\{1, x\}$. The general solution is $A(x) = C_1 + C_2 x$. We need to satisfy the boundary conditions: $0 = A(0) = C_1$. So, $C_1 = 0$. And $0 = A(L) = C_2 \underset{\neq 0}{L} \Rightarrow C_2 = 0$

Thus the only solution of the ODE which satisfies the BCs is the zero solution.

Thus $\lambda = 0$ is NOT an eigenvalue.

<u>Case 3</u> $\lambda < 0$. My style set $\lambda = -\mu^2$ <u>with $\mu > 0$</u>

We need to solve

$$A'' + (-\mu^2)A = 0 \text{ subject to } A(0) = A(L) = 0.$$

$A'' - \mu^2 A = 0$. A fundamental set of solutions is $\{e^{\mu x}, e^{-\mu x}\}$. Another fund. set of solutions is $\{\mathrm{ch}(\mu x), \mathrm{sh}(\mu x)\}$.

The general solution is

$$A(x) = c_1 e^{-\mu x} + c_2 e^{\mu x}.$$

I need $A(x)$ s.t.

$$\boxed{\begin{aligned} 0 &= A(0) = c_1 + c_2 \\ 0 &= A(L) = c_1 e^{-\mu L} + c_2 e^{+\mu L} \end{aligned}}$$

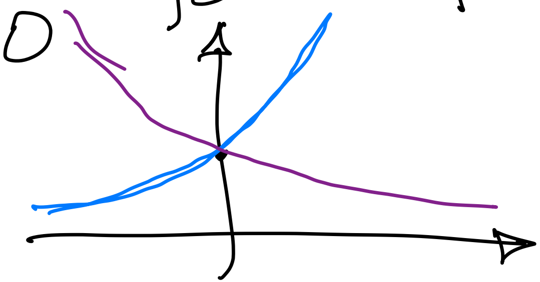
unknowns are $c_1$ & $c_2$.

Math 204 tells me that this homogeneous system has a nontrivial sol. iff $\begin{vmatrix} 1 & 1 \\ e^{-\mu L} & e^{\mu L} \end{vmatrix} = 0$

that is iff $e^{\mu L} - e^{-\mu L} = 0$

or iff $\underline{e^{\mu L}} = \underline{e^{-\mu L}}$

iff $\mu L = 0$ iff $\mu = 0$



Hence, for $\mu > 0$ there is no nontrivial solutions. Since $\mu > 0$,

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 e^{-\mu L} + c_2 e^{\mu L} = 0 \end{cases} \Rightarrow c_1 = c_2 = 0.$$

Thus, no negative eigenvalues.

<u>Conclusion</u> All solutions of

$$-A'' = \lambda A \text{ subject to } A(0)=A(L)=0$$

are $\lambda_m = \left(\frac{m\pi}{L}\right)^2$ with $A_m(x) = \sin\left(\frac{m\pi}{L}x\right)$

with $m \in \mathbb{N} = \{1, 2, \dots\}$

The corresponding $B_m(t) = e^{-\left(\frac{m\pi}{L}\right)^2 \kappa t}$.

Thus, All the functions

$$\boxed{u_m(x,t) = b_m\left(e^{-\left(\frac{m\pi}{L}\right)^2 \kappa t}\right)\sin\left(\frac{m\pi}{L}x\right)}$$

with $\boxed{m \in \mathbb{N} \text{ and } b_m \in \mathbb{R}}$ are sols of

$$\boxed{\begin{aligned}&\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}\\ &u(0,t)=0 \quad u(L,t)=0 \quad \forall t \geq 0\end{aligned}}$$

Since this BVP is linear and homogeneous any linear combination of the above solutions, is again a solution:

$$\forall n \in \mathbb{N} \qquad \sum_{m=1}^{n} b_m \left(e^{-\left(\frac{m\pi}{L}\right)^2 k t}\right) \sin\left(\frac{m\pi}{L}x\right)$$

is a solution of the given BVP.

Now we go back to IC $u(x,0)=f(x)$

$0 \leq x \leq L$

The question is: Can we find $b_m$-s such that

$$(*) \quad \boxed{f(x) = \sum_{m=1}^{n} b_m \sin\left(\frac{m\pi}{L} x\right)}$$

how to calculate $b_m$-s?

The answer is Yes, provided that $f(0)=0$ and $f(L)=0$ (implied by the original problem BCs + IC) and that $f$ is differentiable, and that we allow infinite series. Then for all such $f$'s there exist $b_1, b_2, \dots$ s.t. $(*)$ holds.

The functions $\sin\left(\frac{m\pi}{L}x\right)$, $m \in \mathbb{N}$ are mutually orthogonal in the following sense:

$$\int_0^L \sin\left(\frac{k\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = 0$$

$$\forall k, m \in \mathbb{N} \text{ s.t. } k \neq m.$$

This is 125 problem.

Let us do the following indefinite integral

$$\int \sin(\alpha x)\sin(\beta x)\,dx$$

We use trig. identity:

$$\sin(\alpha x)\sin(\beta x) = \cos((\alpha-\beta)x) - \cos((\alpha+\beta)x)$$

$\cos(u-v) = \cos u \cos v + \sin u \sin v$ ✓

$u=v$

$1$

$\cos(u+v) = \cos u \cos v - \sin u \sin v$

$-$

$2\sin u \sin v = \cos(u-v) - \cos(u+v)$