Calculating Fourier Series and Understanding their Convergence

$$\boxed{f : [-L, L] \to \mathbb{R}}$$

$L > 0$
$L \in \mathbb{R}$

f is piecewise continuous function.

then we can calculate the Fourier coefficients of f

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(\xi)\, d\xi \ , \quad a_k = \frac{1}{L} \int_{-L}^{L} f(\xi) \cos\left(\frac{k\pi}{L}\xi\right) d\xi$$

$$b_k = \frac{1}{L} \int_{-L}^{L} f(\xi) \sin\left(\frac{k\pi}{L}\xi\right) d\xi \ , \quad k \in \mathbb{N}$$

positive integer

Now we can write the Fourier Series corresp. to f:

$$f \sim \boxed{a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi}{L}x\right) + \sum_{k=1}^{+\infty} b_k \sin\left(\frac{k\pi}{L}x\right)}$$

n-th partial sum of the Fourier series

$$S_n^f(x) = \boxed{a_0 + \sum_{k=1}^{n} a_k \cos\left(\frac{k\pi}{L}x\right) + \sum_{k=1}^{n} b_k \sin\left(\frac{k\pi}{L}x\right)}$$

What is $\lim_{n\to\infty} S_n^f(x) = ?$

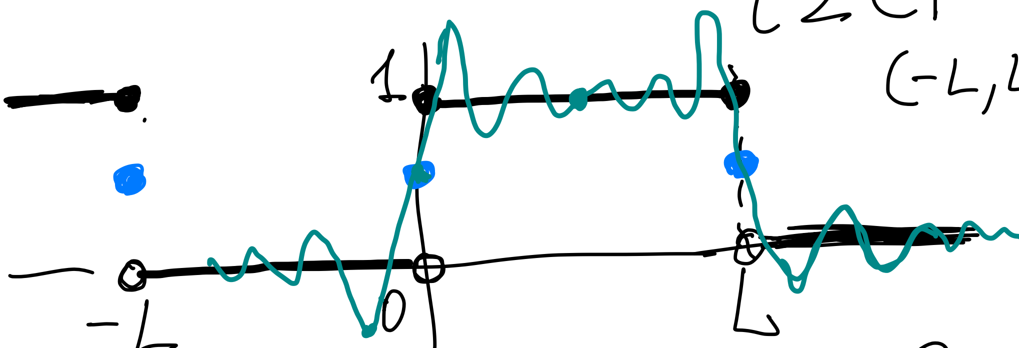
Clearly $S_n^f(x)$ is periodic with period $2L$

Recall that $\tilde{f}$ is the periodic extension of f with period $2L$

$\tilde{f}$ is piecewise continuous.

We define f_{Fourier} by modifying f at the points of its discontinuity. We set

$$\tilde{f}_{\text{Fourier}}(x) = \begin{cases} \tilde{f}(x) & \text{if } \tilde{f} \text{ is continuous at } x \\ \frac{1}{2}\left(\tilde{f}(x+) + \tilde{f}(x-)\right) & \text{if } \tilde{f} \text{ not cont @ } x \end{cases}$$



$[-L, L]$ unit step function restricted to $[-L, L]$

Two Fourier's Convergence Thms:

<u>Pointwise</u> If $\tilde{f}$ is piecewise smooth, then

$$S_n^f(x) \to \tilde{f}_{Fourier}(x) \text{ for all } x \in \mathbb{R}.$$

<u>Uniform</u> If $\tilde{f}$ is piecewise smooth and $\tilde{f}$ is CONTINUOUS, then S_n^f converges to $\tilde{f}_{Fourier}$ UNIFORMLY on $\mathbb{R}$:

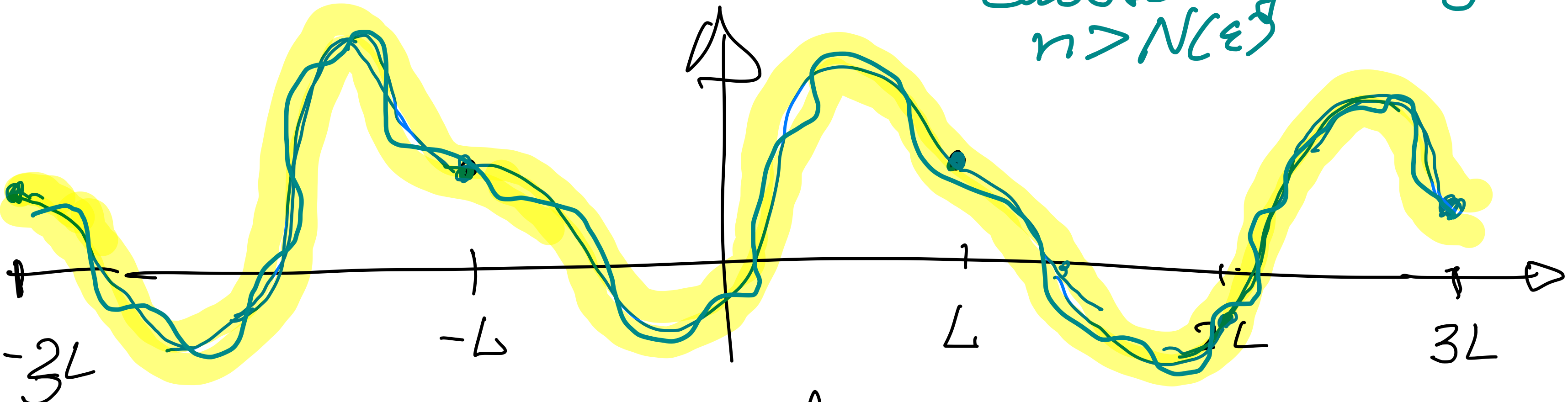
$$\forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbb{R} \text{ such that}$$

$$\forall n \in \mathbb{N} \ n > N(\varepsilon) \Rightarrow \underbrace{\forall x \in \mathbb{R} \ \left|S_n^f(x) - \tilde{f}_{Fou}(x)\right| < \varepsilon}$$

Recall that the period ext f must be cont

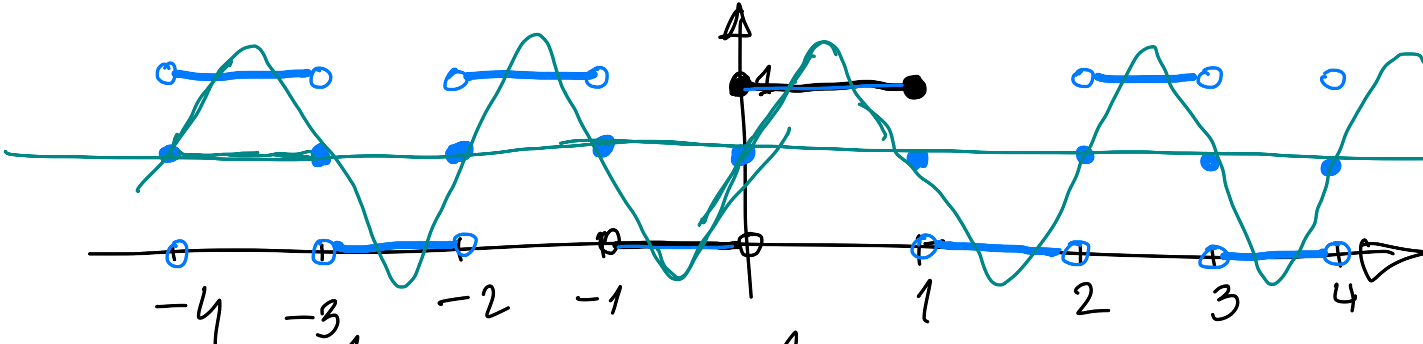
$$\boxed{f_{Fourie}(x)} - \varepsilon < \underbrace{\boxed{S_n^f(x)}} < f_{Fourie}(x) + \varepsilon$$

choose big enough $n > N(\varepsilon)$



The Fourier series for

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ 1 & 0 \le x \le 1 \end{cases}$$



$L=1$

$2L=1$

$$a_0 = \frac{1}{2}\int_{-1}^{1} us(\xi)\,d\xi = \frac{1}{4}\int_0^1 1\,d\xi = \frac{1}{2}$$

$k \in \mathbb{N},$

$$a_k = \frac{1}{1}\int_{-1}^{1} us(\xi)\cos(k\pi\xi)\,d\xi = \int_0^1 \cos(k\pi\xi)\,d\xi = \text{FTC}$$

indefinite integral

$$= \frac{1}{k\pi}\sin(k\pi\xi)\Big|_0^1 = \frac{1}{k\pi}\sin(k\pi) = 0$$

$$b_k = \frac{1}{1}\int_{-1}^{1} us(\xi)\sin(k\pi\xi)\,d\xi = \int_0^1 \sin(k\pi\xi)\,d\xi = -\frac{1}{k\pi}c(k\pi\xi)\Big|_0^1$$

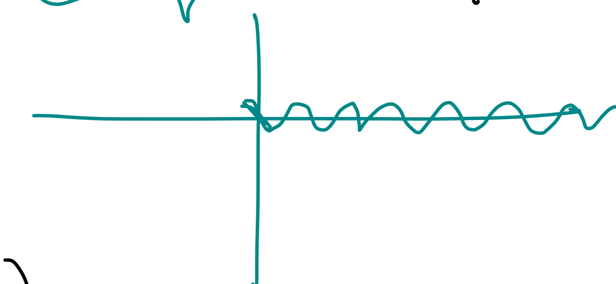
$$= \frac{1}{k\pi}\left(1-(-1)^k\right)$$

$b_k = 0$ k even $\quad b_k = \frac{2}{k\pi}$ k odd

The Fourier series is

$$\frac{1}{2} + \frac{2}{\pi}\sum_{j=1}^{+\infty}\frac{1}{2j-1}\sin\left((2j-1)\pi x\right)$$

$$= \frac{1}{2} + \frac{2}{\pi}\sin(\pi x) + \frac{2}{3\pi}\sin(3\pi x) + \frac{2}{5\pi}\sin(5\pi x) + \cdots$$



The partial sums are

$$\frac{1}{2}$$

$$\frac{1}{2} + \frac{2}{\pi}\sin(\pi x)$$