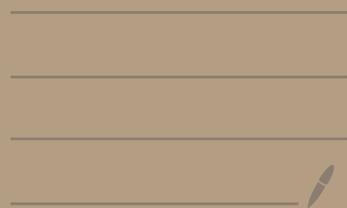


# Integrating Fourier Series



$$f: [-L, L] \rightarrow \mathbb{R}$$

piecewise smooth

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

Fourier

$$F(x) = \int_0^x f(\tau) d\tau$$

This function is continuous on  $[-L, L]$  therefore it coincides with its Fourier extension on  $(-L, L)$

Assume in addition that  $F(-L) = F(L)$ .

This condition is equivalent to  $a_0 = 0$ , that is  $\int_{-L}^L f(\tau) d\tau = 0$ .

$$-\int_{-L}^0 f(\tau) d\tau = \int_0^L f(\tau) d\tau$$

Then the periodic extension of  $F(x)$  is continuous. Then its F.S. converges uniformly.

Now calculate the F.S. of  $F(x)$ .

$$\tilde{F}(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

$A_0$  is "hard".

Integration by parts

$$u = F \quad u'(z) = f(z)$$

$$A_n = \frac{1}{L} \int_{-L}^L \underbrace{F(z)}_u \underbrace{\cos\left(\frac{n\pi}{L}z\right)}_{v'} dz$$

$$v'(z) = \cos\left(\frac{n\pi}{L}z\right)$$

$$v(z) = \frac{L}{n\pi} \sin\left(\frac{n\pi}{L}z\right)$$

$$= \frac{1}{L} \left( \underbrace{\frac{L}{n\pi} F(L) \sin\left(\frac{n\pi}{L}L\right) - \frac{L}{n\pi} F(-L) \sin\left(\frac{n\pi}{L}(-L)\right)}_{=0} + \int_{-L}^L \underbrace{\frac{L}{n\pi} \sin\left(\frac{n\pi}{L}z\right)}_{L b_n} f(z) dz \right)$$

$$= -\frac{L b_n}{n\pi L}$$

$$B_n \stackrel{= \frac{L a_n}{n\pi}}{=} \frac{1}{L} \int_{-L}^L F(z) \sin\left(\frac{n\pi}{L}z\right) dz = \frac{1}{L} \left( \underbrace{\left[ -F(z) \frac{L}{n\pi} \cos\left(\frac{n\pi}{L}z\right) \right]_{-L}^L}_{\substack{F(-L) \frac{L}{n\pi} (-1)^{n_0} \\ - F(L) \frac{L}{n\pi} (-1)^n}} + \frac{L}{n\pi} \int_{-L}^L f(z) dz \right) \stackrel{= L a_n}{}$$

Thus we have  $\frac{1}{2L} \int_{-L}^L F(x) dx$  is the average of  $F(x)$  on  $[-L, L]$

$$\tilde{F}(x) = A_0 - \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{b_n}{n} \cos\left(\frac{n\pi}{L}x\right) + \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{a_n}{n} \sin\left(\frac{n\pi}{L}x\right)$$

all  $x$ , uniformly

$$x=0, F(0) = \tilde{F}(0) = 0$$

$$0 = A_0 - \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{b_n}{n}$$

$$A_0 = \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{b_n}{n}$$

What we learn from  $\square$  is that the Fourier Coefficients of  $\tilde{F}$  converge to 0

faster than the Fourier Coeff. of  $f$ .

Thus increasing smoothness, that is what  $\int_0^x$  does makes the Fourier Coeffs converge to 0 faster.

Now integrate term by term

recall, we assume  $a_0=0$

$$a_n \int_0^x \cos\left(\frac{n\pi}{L}z\right) dz = \frac{L a_n}{n\pi} \sin\left(\frac{n\pi}{L}z\right) \Big|_0^x = \frac{L a_n}{n\pi} \sin\left(\frac{n\pi}{L}x\right)$$

$$b_n \int_0^x \sin\left(\frac{n\pi}{L}z\right) dz = \frac{L b_n}{n\pi} \left( -\cos\left(\frac{n\pi}{L}z\right) \Big|_0^x \right) = \frac{L b_n}{n\pi} \left( 1 - \cos\left(\frac{n\pi}{L}x\right) \right)$$

Thus we get

$$\frac{L}{\pi} \sum_{n=1}^{\infty} \frac{b_n}{n} - \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{b_n}{n} \cos\left(\frac{n\pi}{L}x\right) + \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{a_n}{n} \sin\left(\frac{n\pi}{L}x\right)$$

The F. series  $\square$  is identical to the F. series  $\square$

Hence, the term by term integration of the Fourier series  $\square$  of  $f(x)$  provided that  $a_0 = 0$ , will result in the Fourier series of the function  $F(x) = \int_0^x f(\xi) d\xi$ . Since  $a_0 = 0$  is equivalent to  $F(-L) = F(L)$ , we have that the periodic extension  $\tilde{F}(x)$  of  $F(x)$  is CONTINUOUS. Therefore the Fourier series  $\square = \square$  converges uniformly to  $\tilde{F}(x)$ .