Integrating Fourier Series

$f : [-L, L] \to \mathbb{R}$ piecewise smooth

$$\boxed{\underbrace{\tilde{f}_{\text{Fourier}}}(x) \equiv a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} x\right)}$$

$$F(x) = \int_0^x f(\xi)\, d\xi$$

This function is continuous on $[-L, L]$ therefore is coincides with its Fourier extension on $(-L, L)$

Assume in addition that $F(-L) = F(L)$.

$$-\int_{-L}^{0} f(\xi)\, d\xi = \int_0^L f(\xi)\, d\xi$$

This condition is equivalent to $a_0 = 0$, that is $\int_{-L}^{L} f(\xi)\, d\xi = 0$.

Then the periodic extension of $F(x)$ is continuous. Then its F.S. converges uniformly

Now calculate the F.S. of $F(x)$.

$$\tilde{F}(x) = A_0 + \sum_{n=1}^{\infty} \underbrace{A_n}_{=-\frac{Lb_n}{n\pi}} \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} \underbrace{B_n}_{\frac{La_n}{n\pi}} \sin\left(\frac{n\pi}{L}x\right)$$

A_0 is "hard".

Integration by parts

$u = F$

$u'(\xi) = f(\xi)$

$v'(\xi) = \cos\left(\frac{n\pi}{L}\xi\right)$

$v(\xi) = \frac{L}{n\pi}\sin\left(\frac{n\pi}{L}\xi\right)$

$$A_n = \frac{1}{L}\int_{-L}^{L} \underbrace{F(\xi)}_{u} \underbrace{\cos\left(\frac{n\pi}{L}\xi\right)}_{v'} d\xi \overset{*}{=}$$

$$= \frac{1}{L}\left(\underbrace{\frac{L}{n\pi}F(L)\sin\left(\frac{n\pi}{L}L\right) - \frac{L}{n\pi}F(-L)\sin\left(\frac{n\pi}{L}(-L)\right)}_{=0} - \int_{-L}^{L} \frac{L}{n\pi} \underbrace{\sin\left(\frac{n\pi}{L}\xi\right)f(\xi)d\xi}_{Lb_n}\right)$$

$$= -\frac{Lb_n}{n\pi}$$

$$B_n = \frac{1}{L}\int_{-L}^{L} F(\xi)\sin\left(\frac{n\pi}{L}\xi\right)d\xi = \frac{1}{L}\left(\underbrace{\left.-F(\xi)\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}\xi\right)\right|_{-L}^{L}}_{F(-L)\frac{L}{n\pi}(-1)^n - F(L)\frac{L}{n\pi}(-1)^n} + \frac{L}{n\pi}\int_{-L}^{L} \underbrace{f(\xi)c}_{La_n}\right)$$

$$B_n = \frac{La_n}{n\pi}$$

Thus we have is the average of $F(x)$ on $[-L,L]$

$$\boxed{\tilde{F}(x) = A_0 - \frac{L}{\pi}\sum_{n=1}^{\infty} \frac{b_n}{n}\cos\left(\frac{n\pi}{L}x\right) + \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n}\sin\left(\frac{n\pi}{L}x\right)}$$

all x, uniformly

$x = 0$, $F(0) = \tilde{F}(0) = 0$

$$0 = A_0 - \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{b_n}{n}$$

$$A_0 = \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{b_n}{n}.$$

What we learn from $\square$ is that the Fourier Coefficients of $\tilde{F}$ converge to 0 faster than the Fourier Coeff. of f.

Thus increasing smoothness, that is what $\int_0^x$ does makes the Fourier Coeffs converge to 0 faster.

Now integrate term by term

recall, we assume $a_0 = 0$

$$a_n \int_0^x \cos\left(\frac{n\pi}{L}\xi\right) d\xi = \frac{La_n}{n\pi} \sin\left(\frac{n\pi}{L}\xi\right)\Big|_0^x = \frac{La_n}{n\pi} \sin\left(\frac{n\pi}{L}x\right)$$

$$b_n \int_0^x \sin\left(\frac{n\pi}{L}\xi\right) d\xi = \frac{Lb_n}{n\pi}\left(-\cos\left(\frac{n\pi}{L}\xi\right)\Big|_0^x\right) = \frac{Lb_n}{n\pi}\left(1-\cos\left(\frac{n\pi}{L}x\right)\right)$$

Thus we get $\frac{L}{\pi}\sum_{n=1}^{\infty}\frac{b_n}{n} - \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{b_n}{n}\cos\left(\frac{n\pi}{L}x\right) + \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n}\sin\left(\frac{n\pi}{L}x\right)$

The F. series is identical to the F. series

Hence, the term by term integration of the Fourier series of $f(x)$), provided that $a_0 = 0$, will result in the Fourier series of the function $F(x) = \int_0^x f(\xi)\,d\xi$. Since $a_0 = 0$ is equivalent to $F(-L) = F(L)$, we have that the periodic extension $\tilde{F}(x)$ of $F(x)$ is CONTINUOUS. Therefore the Fourier series $=$ converges uniformly to $\tilde{F}(x)$.