AXIOMS OF A VECTOR SPACE

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ABSTRACT. Did I get it right?

Definition 1. A subset $\mathbb{F}$ of $\mathbb{C}$ is called a *scalar field* if the following five statements hold.

SF1 $0, 1 \in \mathbb{F}$.
SF2 If $\alpha, \beta \in \mathbb{F}$, then $\alpha + \beta \in \mathbb{F}$ and $\alpha\beta \in \mathbb{F}$.
SF3 If $\alpha \in \mathbb{F}$, then $-\alpha \in \mathbb{F}$.
SF4 If $\alpha \in \mathbb{F}$ and $\alpha \neq 0$, then $\dfrac{1}{\alpha} \in \mathbb{F}$.
SF5 If $\alpha \in \mathbb{F}$, then $\overline{\alpha} \in \mathbb{F}$.

Definition 2. A set $\mathcal{V}$ is called a *commutative group* if the following five statements hold.

CG1 There exists a mapping $+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$.

(The mapping in CG1 is called *addition* and its value on a pair $(u, v) \in \mathcal{V} \times \mathcal{V}$ is denoted by $u + v$.)

CG2 For all $u, v, w \in \mathcal{V}$ we have $u + (v + w) = (u + v) + w$.
CG3 For all $u, v \in \mathcal{V}$ we have $u + v = v + u$.
CG4 There exists an element $0_{\mathcal{V}} \in \mathcal{V}$ such that $v + 0_{\mathcal{V}} = v$ for all $v \in \mathcal{V}$.
CG5 For each $v \in \mathcal{V}$ there exists $w \in \mathcal{V}$ such that $v + w = 0_{\mathcal{V}}$.

It can be proved that for a given $v \in \mathcal{V}$ the vector $w \in \mathcal{V}$ whose existence is postulated by CG5 is unique. This vector is called the *opposite* of v and it is denoted by $-v$.

Definition 3. A commutative group $\mathcal{V}$ with the addition $+$ is called a *vector space over a scalar field* $\mathbb{F}$ if the following five statements hold.

VS1 There exists a mapping $\cdot : \mathbb{F} \times \mathcal{V} \to \mathcal{V}$.

(The mapping in VS1 is called *scaling* and its value on a pair $(\alpha, v) \in \mathbb{F} \times \mathcal{V}$ is denoted by $\alpha \cdot v$, or simply αv.)

VS2 For all $\alpha, \beta \in \mathbb{F}$ and all $v \in \mathcal{V}$ we have $\alpha(\beta v) = (\alpha\beta)v$.
VS3 For all $\alpha, \beta \in \mathbb{F}$ and all $v \in \mathcal{V}$ we have $(\alpha + \beta)v = \alpha v + \beta v$.
VS4 For all $\alpha \in \mathbb{F}$ and all $u, v \in \mathcal{V}$ we have $\alpha(u + v) = \alpha u + \alpha v$.
VS5 For all $v \in \mathcal{V}$ we have $1v = v$.

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