MATH 504 Assignment 2 November 14, 2011 Name ____________________

Problem 1. Let $(\mathcal{V}, \langle \cdot\,, \cdot \rangle)$ be an inner product space over a scalar field $\mathbb{F}$. Let $\|\cdot\|$ be the corresponding norm on $\mathcal{V}$. That is, for $v \in \mathcal{V}$, $\|v\| := \sqrt{\langle v, v\rangle}$. Find a necessary and sufficient condition (in terms of the vectors $v_1, \ldots, v_k \in \mathcal{V}$) for the following equality

$$\|v_1 + \cdots + v_k\| = \|v_1\| + \cdots + \|v_k\|.$$

Problem 2. Let $\mathcal{V}$ be a finite dimensional vector space and let $T : \mathcal{V} \longrightarrow \mathcal{V}$ be a linear map. Put $T^0 = I$, $T^1 = T$, and $T^j = T^{j-1} \circ T$, for $j \in \mathbb{N}$.

(a) Prove that there exists $k \in \mathbb{N}$ such that $\mathcal{N}\big(T^k\big) = \mathcal{N}\big(T^{k+1}\big)$.

(b) Prove that $\mathcal{N}\big(T^k\big) = \mathcal{N}\big(T^l\big)$ for each $l \in \mathbb{N}$, $l > k$ where k is a number from (a).

(c) Explore the situation with $\mathcal{R}\big(T^j\big)$ with $j \in \mathbb{N}$. Formulate your statements and prove them.

Problem 3. Let $\mathcal{V}$ be a finite dimensional vector space over a scalar field $\mathbb{F}$ and $n = \dim \mathcal{V}$. Let $v_1, v_2, \ldots, v_n$ be a basis of $\mathcal{V}$ and let $\langle \cdot\,, \cdot \rangle$ be an inner product on $\mathcal{V}$. Suppose that $x_1, x_2, \ldots, x_n$ are arbitrary scalars in $\mathbb{F}$. Prove that there exists a vector $v \in \mathcal{V}$, such that

$$\langle v, v_j\rangle \;=\; x_j \quad \text{for all} \quad j \in \{1, \ldots, n\}.$$

Problem 4. Let $\mathbb{C}[z]$ be the set of all polynomials with complex coefficients. Then $\mathbb{C}[z]$ is a vector space over $\mathbb{C}$. (You do not need to prove this.) By $D : \mathcal{P} \to \mathcal{P}$ we denote the differentiation operator

$$\big(Df\big)(z) = f'(z), \quad f \in \mathbb{C}[z].$$

Let $\mathcal{Q}$ be a nontrivial finite dimensional subspace of $\mathbb{C}[z]$ which is invariant under D; that is such that $D\mathcal{Q} \subseteq \mathcal{Q}$. Prove that there exist $n \in \mathbb{N} \cup \{0\}$ such that

$$\mathcal{Q} = \big\{ f \in \mathbb{C}[z] : \deg f \leq n \big\}.$$

Problem 5. Let $\mathcal{V}$ be a finite dimensional vector space over a field $\mathbb{F}$. Let $\langle \cdot\,, \cdot \rangle$ be an inner product on $\mathcal{V}$. Let x and y be fixed nonzero vectors in $\mathcal{V}$. Define the mapping $T \in \mathcal{L}(\mathcal{V})$ by

$$Tv = v - \langle v, x\rangle y, \qquad v \in \mathcal{V}.$$

You do not need to prove that $T \in \mathcal{L}(\mathcal{V})$. Answer the following questions and provide complete rigorous justifications.

(a) Determine all eigenvalues and the corresponding eigenspaces of T.

(b) Determine an explicit formula for T^*.

(c) Describe all mappings Q on $\mathcal{V}$ for which $TQ = QT$.

(d) Determine a necessary and sufficient condition for T to be normal.

(e) Determine a necessary and sufficient condition for T to be self-adjoint.