MATH 504 Examination 1 October 26, 2011

Name ______________________

Problem 1. State and prove the Steinitz exchange lemma.

Problem 2. Let $\mathcal{V}$ be a finite dimensional vector space over $\mathbb{C}$ and let $T \in \mathcal{L}(\mathcal{V})$. Prove that there exists a basis $\mathcal{B}$ of $\mathcal{V}$ such that $\mathsf{M}_{\mathcal{B}}(T)$ is an upper triangular matrix.

Do two out of three problems below.

Problem 3. Let $\mathcal{V}$ be finite-dimensional vector space over $\mathbb{F}$. Let $\mathcal{U}$ be a subspace of $\mathcal{V}$. Set $k = \dim \mathcal{U}, m = \dim \mathcal{V}$. Consider the following set

$$\mathcal{K} = \bigl\{T \in \mathcal{L}(\mathcal{V}) \, : \, T\mathcal{U} \subseteq \mathcal{U}\bigr\}.$$

Prove that $\mathcal{K}$ is a subspace of $\mathcal{L}(\mathcal{V})$. (This is easy, but do it right.) Determine $\dim \mathcal{K}$. A formal proof is required for full credit.

Problem 4. Let $\mathbb{C}[z]$ be a vector space of polynomials over $\mathbb{C}$. Let $q \in \mathbb{C}[z]$ be a fixed nonzero polynomial and let $z_0 \in \mathbb{C}$ be a fixed complex number. Define a mapping T on $\mathbb{C}[z]$ by

$$Tp = p - p(z_0)q, \quad p \in \mathbb{C}[z].$$

Then $T \in \mathcal{L}\bigl(\mathbb{C}[z]\bigr)$. (You don't need to prove this.)

(a) Under some condition on the polynomial q and the number z_0 the mapping T is invertible. Discover this condition; state it and prove your claim.

(b) Assume that the condition you stated in (a is satisfied. Find the formula for the inverse of T.

(c) Assume that the condition you stated in (a is not satisfied. Find $\mathcal{N}(T)$.

(d) Find the eigenvalues and eigenspaces of T. They should be given in terms of the polynomial q and the number z_0.

Problem 5. Assume

- k is a natural number,
- $\mathcal{V}$ is a vector space over $\mathbb{F}$,
- $T \in \mathcal{L}(\mathcal{V})$,
- $\lambda_1, \ldots \lambda_k$ are mutually distinct scalars in $\mathbb{F}$,
- $v_1, \ldots, v_k \in \mathcal{V}$,
- $Tv_j = \lambda_j v_j, \ j \in \{1, \ldots, k\}$,
- $\mathcal{W}$ is a subspace of $\mathcal{V}$ which is invariant under T, that is $T\mathcal{W} \subseteq \mathcal{W}$.

Prove the following implication:

$$\text{If } v_1 + \cdots + v_k \in \mathcal{W}, \text{ then } v_j \in \mathcal{W} \ \text{ for all } \ j \in \{1, \ldots, k\}.$$