MATH 504 Assignment 1
October 15, 2013 Name ______________________

Problem 1. Let D be a finite set and let $\mathbb{F}$ be a scalar field. Then the set of all functions defined on D with values in $\mathbb{F}$ is a vector space over $\mathbb{F}$ with the addition and scalar multiplication of functions defined pointwise. This space is denoted by $\mathbb{F}^D$.

(a) Prove that $\mathbb{F}^D$ is finite dimensional if and only if D is finite.

(b) If D is finite, then $\dim\bigl(\mathbb{F}^D\bigr) = |D|$.

Problem 2. Consider the vector space $\mathbb{R}^\mathbb{R}$ of all real valued functions defined on $\mathbb{R}$. This vector space is considered over the field $\mathbb{R}$. The purpose of this exercise is to study some special subspaces of the vector space $\mathbb{R}^\mathbb{R}$. Let γ be an arbitrary (fixed) real number. Consider the set

$$\mathcal{S}_\gamma := \Bigl\{ f \in \mathbb{R}^\mathbb{R} : \exists\, a, b \in \mathbb{R} \ \text{ such that } \ f(t) = a \sin(\gamma t + b) \ \ \forall\, t \in \mathbb{R} \Bigr\}.$$

(a) Do you see exceptional values for γ for which the set $\mathcal{S}_\gamma$ is particularly simple?

(b) Prove that $\mathcal{S}_\gamma$ is a subspace of $\mathbb{R}^\mathbb{R}$.

(c) For each $\gamma \in \mathbb{R}$ find a basis for $\mathcal{S}_\gamma$. Plot the function $\gamma \mapsto \dim \mathcal{S}_\gamma$.

Problem 3. Let $\mathcal{V}$ be a vector space over $\mathbb{F}$. Let $\mathcal{A}$ be a linearly independent subset of $\mathcal{V}$. Let $u \in \mathcal{V}$ be arbitrary. By $u + \mathcal{A}$ we denote the set of vectors $\{u + v : v \in \mathcal{A}\}$.

(a) Prove the following implication. If $w \notin \operatorname{span} \mathcal{A}$, then $w + \mathcal{A}$ is a linearly independent set.

(b) Is the converse of the implication in (a) true?

(c) Let $\alpha_1, \cdots, \alpha_n \in \mathbb{F}$, let $v_1, \ldots, v_n$ be distinct vectors in $\mathcal{A}$ and let $w = \alpha_1 v_1 + \cdots + \alpha_n v_n$. Find a necessary and sufficient condition (in terms of $\alpha_1, \ldots, \alpha_n$) for the linear independence of the vectors $v_1 + w, \ldots, v_n + w$.

Problem 4. Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over a scalar field $\mathbb{F}$. Assume that $\dim \mathcal{V} = m$ and $\dim \mathcal{W} = n$. Let $T : \mathcal{V} \to \mathcal{W}$ be a linear mapping. Prove that we can choose a basis $\mathcal{B}$ of $\mathcal{V}$ and a basis $\mathcal{C}$ of $\mathcal{W}$ such that for some integer k, $0 \le k \le \min\{m, n\}$ we have

$$\mathsf{M}_{\mathcal{C}}^{\mathcal{B}}(T) = \begin{bmatrix} \mathsf{I}_{k\times k} & \mathsf{0}_{k\times(m-k)} \\ \mathsf{0}_{(n-k)\times k} & \mathsf{0}_{(n-k)\times(m-k)} \end{bmatrix}.$$

Problem 5. Let $\mathcal{V}$ be a finite dimensional vector space over a field $\mathbb{F}$ and let $T : \mathcal{V} \to \mathcal{V}$ be a linear mapping. Put $T^2 := T \circ T$.

(a) Discover and prove an inclusion relation between $\operatorname{nul}(T)$ and $\operatorname{nul}(T^2)$, and between $\operatorname{ran}(T)$ and $\operatorname{ran}(T^2)$.

(b) If $\operatorname{ran}(T) = \operatorname{ran}(T^2)$, prove that $(\operatorname{ran}(T)) \cap (\operatorname{nul}(T)) = \{0_\mathcal{V}\}$.

(c) If $\operatorname{ran}(T) = \operatorname{ran}(T^2)$, prove that $\mathcal{V}$ is a direct sum of $\operatorname{ran}(T)$ and $\operatorname{nul}(T)$.

Problem 6. Let $\mathcal{V}$ be a vector space over $\mathbb{F}$ and $T \in \mathcal{L}(\mathcal{V})$. Assume that that there exists a function $f : \mathcal{V} \to \mathbb{F}$ such that $Tv = f(v)\, v$ for each $v \in \mathcal{V}$. Prove that T is a multiple of the identity mapping, that is, there exists $\alpha \in \mathbb{F}$ such that $Tv = \alpha\, v$ for each $v \in \mathcal{V}$. (A plain English explanation: The equation $Tv = f(v)\, v$ is telling us that T scales each vector in $\mathcal{V}$ by the scaling coefficient $f(v)$. The point of the problem is to prove that T must scale each vector by the same coefficient. This is a consequence of the linearity of T.)

Problem 7. Let $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ be nontrivial finite dimensional vector spaces over a scalar field $\mathbb{F}$.

(a) Let $S \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ be an arbitrary fixed operator. Define

$$\mathbf{S} : \mathcal{L}(\mathcal{V}, \mathcal{W}) \to \mathcal{L}(\mathcal{U}, \mathcal{W}) \qquad \text{by} \qquad \mathbf{S}(T) = TS, \quad T \in \mathcal{L}(\mathcal{V}, \mathcal{W}).$$

Prove that $\mathbf{S}$ is injective iff S is surjective. Prove that $\mathbf{S}$ is surjective iff S is injective.

(b) Let $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ be an arbitrary fixed operator. Define

$$\mathbf{T} : \mathcal{L}(\mathcal{U}, \mathcal{V}) \to \mathcal{L}(\mathcal{U}, \mathcal{W}) \qquad \text{by} \qquad \mathbf{T}(S) = TS, \quad S \in \mathcal{L}(\mathcal{U}, \mathcal{V}).$$

In (a) we characterized injectivity and surjectivity of $\mathbf{S}$. Formulate and prove an analogous statement for $\mathbf{T}$.