MATH 504 Assignment 2 November 4, 2013 Name ____________

Problem 1. Let $\mathcal{V}$ be a finite dimensional vector space and let $T : \mathcal{V} \longrightarrow \mathcal{V}$ be a linear map. Put $T^0 = I$, $T^1 = T$, and $T^j = T^{j-1} \circ T$, for $j \in \mathbb{N}$.

(a) Prove that there exists $k \in \mathbb{N}$ such that $\operatorname{nul}(T^k) = \operatorname{nul}(T^{k+1})$.

(b) For k from (a) we have $\operatorname{nul}(T^k) = \operatorname{nul}(T^l)$ for each $l \in \mathbb{N}$, $l > k$.

(c) Explore $\operatorname{ran}(T^j)$ with $j \in \mathbb{N}$ in the spirit of the (a) and (b). Formulate your statements and prove them.

Problem 2. Let $\mathbb{C}[z]$ be the set of all polynomials with complex coefficients. For $n \in \mathbb{N}$ by $\mathbb{C}[z]_{<n}$ we denote the complex vector subspace of $\mathbb{C}[z]$ of all polynomials whose degree is less than n. (You do not need to prove this.) By $D : \mathbb{C}[z] \to \mathbb{C}[z]$ we denote the differentiation operator

$$\big(Df\big)(z) = f'(z), \quad f \in \mathbb{C}[z].$$

Let $\mathcal{Q}$ be a nontrivial finite dimensional subspace of $\mathbb{C}[z]$. Then $D\mathcal{Q} \subseteq \mathcal{Q}$ if and only if there exists $n \in \mathbb{N}$ such that $\mathcal{Q} = \mathbb{C}[z]_{<n}$.

Problem 3. Let $m \in \mathbb{N}$, let $z_1, \ldots, z_m \in \mathbb{C}$ be distinct complex numbers and let $l_1, \ldots, l_m \in \mathbb{N}$. Set $n = \sum_{j=1}^{m} l_j$. By $\mathbb{C}[z]_{<n}$ we denote the complex vector space of all polynomials with coefficients in $\mathbb{C}$ whose degree is less than n. Prove that the function $T : \mathbb{C}[z]_{<n} \to \mathbb{C}^n$ defined by

$$Tp = \big[p(z_1) \ \cdots \ p^{(l_1-1)}(z_1) \ \cdots \ p(z_m) \ \cdots \ p^{(l_m-1)}(z_m)\big]^\top, \qquad p \in \mathbb{C}[z]_{<n},$$

is an isomorphism. In the definition of T, for $p \in \mathbb{C}[z]_{<n}$ and $j \in \mathbb{N}$, $p^{(j)}$ denotes the jth derivative of p.

Problem 4. Let $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ be an inner product space over a scalar field $\mathbb{F}$ with a positive definite inner product $\langle \cdot, \cdot \rangle$. Let $\| \cdot \|$ be the corresponding norm on $\mathcal{V}$. That is, for $v \in \mathcal{V}$, $\|v\| := \sqrt{\langle v, v \rangle}$. Find a necessary and sufficient condition (in terms of the vectors $v_1, \ldots, v_k \in \mathcal{V}$) for the following equality

$$\|v_1 + \cdots + v_k\| = \|v_1\| + \cdots + \|v_k\|.$$

Problem 5. Let $\mathcal{V}$ be a finite dimensional vector space over a scalar field $\mathbb{F}$ and $n = \dim \mathcal{V}$. Let $v_1, v_2, \ldots, v_n$ be a basis of $\mathcal{V}$ and let $\langle \cdot, \cdot \rangle$ be a positive definite inner product on $\mathcal{V}$. Suppose that $x_1, x_2, \ldots, x_n$ are arbitrary scalars in $\mathbb{F}$. Prove that there exists a vector $v \in \mathcal{V}$, such that

$$\langle v, v_j \rangle \ = \ x_j \quad \text{for all} \quad j \in \{1, \ldots, n\}.$$

Problem 6. Let $\mathcal{V}$ be a finite dimensional vector space over a scalar field $\mathbb{F}$. Assume that $\dim \mathcal{V} > 1$. Let $\langle \cdot, \cdot \rangle$ be a positive definite inner product on $\mathcal{V}$. Let x and y be fixed nonzero vectors in $\mathcal{V}$. Define the operator $T \in \mathcal{L}(\mathcal{V})$ by

$$Tv = v - \langle v, x \rangle y, \qquad v \in \mathcal{V}.$$

You do not need to prove that $T \in \mathcal{L}(\mathcal{V})$. Answer the following questions and provide complete rigorous justifications.

(a) Determine all eigenvalues and the corresponding eigenspaces of T. Provide a proof that you indeed found <u>all</u> the eigenvalues.

(b) Determine an explicit formula for T^*.

(c) Describe all operators Q on $\mathcal{V}$ for which $TQ = QT$.

(d) Determine a necessary and sufficient condition for T to be normal.

(e) Determine a necessary and sufficient condition for T to be self-adjoint.