CHAPTER 2

# *Finite-Dimensional Vector Spaces*

In the last chapter we learned about vector spaces. Linear algebra focuses not on arbitrary vector spaces, but on finite-dimensional vector spaces, which we introduce in this chapter. Here we will deal with the key concepts associated with these spaces: span, linear independence, basis, and dimension.

Let's review our standing assumptions:

> Recall that $\mathbf{F}$ denotes $\mathbf{R}$ or $\mathbf{C}$.
> Recall also that $V$ is a vector space over $\mathbf{F}$.

## *Span and Linear Independence*

A ***linear combination*** of a list $(v_1, \ldots, v_m)$ of vectors in $V$ is a vector of the form

**2.1** $$a_1 v_1 + \cdots + a_m v_m,$$

where $a_1, \ldots, a_m \in \mathbf{F}$. The set of all linear combinations of $(v_1, \ldots, v_m)$ is called the ***span*** of $(v_1, \ldots, v_m)$, denoted $\operatorname{span}(v_1, \ldots, v_m)$. In other words,

*Some mathematicians use the term **linear span**, which means the same as span.*

$$\operatorname{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \ldots, a_m \in \mathbf{F}\}.$$

As an example of these concepts, suppose $V = \mathbf{F}^3$. The vector $(7, 2, 9)$ is a linear combination of $\big((2, 1, 3), (1, 0, 1)\big)$ because

$$(7, 2, 9) = 2(2, 1, 3) + 3(1, 0, 1).$$

Thus $(7, 2, 9) \in \operatorname{span}\big((2, 1, 3), (1, 0, 1)\big)$.

You should verify that the span of any list of vectors in $V$ is a subspace of $V$. To be consistent, we declare that the span of the empty list $()$ equals $\{0\}$ (recall that the empty set is not a subspace of $V$).

If $(v_1, \ldots, v_m)$ is a list of vectors in $V$, then each $v_j$ is a linear combination of $(v_1, \ldots, v_m)$ (to show this, set $a_j = 1$ and let the other $a$'s in 2.1 equal 0). Thus $\operatorname{span}(v_1, \ldots, v_m)$ contains each $v_j$. Conversely, because subspaces are closed under scalar multiplication and addition, every subspace of $V$ containing each $v_j$ must contain $\operatorname{span}(v_1, \ldots, v_m)$. Thus the span of a list of vectors in $V$ is the smallest subspace of $V$ containing all the vectors in the list.

If $\operatorname{span}(v_1, \ldots, v_m)$ equals $V$, we say that $(v_1, \ldots, v_m)$ ***spans*** $V$. A vector space is called ***finite dimensional*** if some list of vectors in it spans the space. For example, $\mathbf{F}^n$ is finite dimensional because

*Recall that by definition every list has finite length.*

$$\big((1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\big)$$

spans $\mathbf{F}^n$, as you should verify.

Before giving the next example of a finite-dimensional vector space, we need to define the degree of a polynomial. A polynomial $p \in \mathcal{P}(\mathbf{F})$ is said to have ***degree*** $m$ if there exist scalars $a_0, a_1, \ldots, a_m \in \mathbf{F}$ with $a_m \neq 0$ such that

**2.2** $$p(z) = a_0 + a_1 z + \cdots + a_m z^m$$

for all $z \in \mathbf{F}$. The polynomial that is identically 0 is said to have degree $-\infty$.

For $m$ a nonnegative integer, let $\mathcal{P}_m(\mathbf{F})$ denote the set of all polynomials with coefficients in $\mathbf{F}$ and degree at most $m$. You should verify that $\mathcal{P}_m(\mathbf{F})$ is a subspace of $\mathcal{P}(\mathbf{F})$; hence $\mathcal{P}_m(\mathbf{F})$ is a vector space. This vector space is finite dimensional because it is spanned by the list $(1, z, \dots, z^m)$; here we are slightly abusing notation by letting $z^k$ denote a function (so $z$ is a dummy variable).

A vector space that is not finite dimensional is called ***infinite dimensional***. For example, $\mathcal{P}(\mathbf{F})$ is infinite dimensional. To prove this, consider any list of elements of $\mathcal{P}(\mathbf{F})$. Let $m$ denote the highest degree of any of the polynomials in the list under consideration (recall that by definition a list has finite length). Then every polynomial in the span of this list must have degree at most $m$. Thus our list cannot span $\mathcal{P}(\mathbf{F})$. Because no list spans $\mathcal{P}(\mathbf{F})$, this vector space is infinite dimensional.

*Infinite-dimensional vector spaces, which we will not mention much anymore, are the center of attention in the branch of mathematics called **functional analysis**. Functional analysis uses tools from both analysis and algebra.*

The vector space $\mathbf{F}^\infty$, consisting of all sequences of elements of $\mathbf{F}$, is also infinite dimensional, though this is a bit harder to prove. You should be able to give a proof by using some of the tools we will soon develop.

Suppose $v_1, \dots, v_m \in V$ and $v \in \operatorname{span}(v_1, \dots, v_m)$. By the definition of span, there exist $a_1, \dots, a_m \in \mathbf{F}$ such that

$$v = a_1 v_1 + \cdots + a_m v_m.$$

Consider the question of whether the choice of $a$'s in the equation above is unique. Suppose $\hat{a}_1, \dots, \hat{a}_m$ is another set of scalars such that

$$v = \hat{a}_1 v_1 + \cdots + \hat{a}_m v_m.$$

Subtracting the last two equations, we have

$$0 = (a_1 - \hat{a}_1) v_1 + \cdots + (a_m - \hat{a}_m) v_m.$$

Thus we have written 0 as a linear combination of $(v_1, \dots, v_m)$. If the only way to do this is the obvious way (using 0 for all scalars), then each $a_j - \hat{a}_j$ equals 0, which means that each $a_j$ equals $\hat{a}_j$ (and thus the choice of $a$'s was indeed unique). This situation is so important that we give it a special name—linear independence—which we now define.

A list $(v_1, \dots, v_m)$ of vectors in $V$ is called ***linearly independent*** if the only choice of $a_1, \dots, a_m \in \mathbf{F}$ that makes $a_1 v_1 + \cdots + a_m v_m$ equal 0 is $a_1 = \cdots = a_m = 0$. For example,

$$\big((1,0,0,0),(0,1,0,0),(0,0,1,0)\big)$$

is linearly independent in $\mathbf{F}^4$, as you should verify. The reasoning in the previous paragraph shows that $(v_1,\dots,v_m)$ is linearly independent if and only if each vector in $\operatorname{span}(v_1,\dots,v_m)$ has only one representation as a linear combination of $(v_1,\dots,v_m)$.

*Most linear algebra texts define linearly independent sets instead of linearly independent lists. With that definition, the set* $\{(0,1),(0,1),(1,0)\}$ *is linearly independent in* $\mathbf{F}^2$ *because it equals the set* $\{(0,1),(1,0)\}$. *With our definition, the list* $\big((0,1),(0,1),(1,0)\big)$ *is not linearly independent (because* $1$ *times the first vector plus* $-1$ *times the second vector plus* $0$ *times the third vector equals* $0$*). By dealing with lists instead of sets, we will avoid some problems associated with the usual approach.*

For another example of a linearly independent list, fix a nonnegative integer $m$. Then $(1,z,\dots,z^m)$ is linearly independent in $\mathcal{P}(\mathbf{F})$. To verify this, suppose that $a_0,a_1,\dots,a_m\in\mathbf{F}$ are such that

**2.3** $$a_0+a_1z+\cdots+a_mz^m=0$$

for every $z\in\mathbf{F}$. If at least one of the coefficients $a_0,a_1,\dots,a_m$ were nonzero, then 2.3 could be satisfied by at most $m$ distinct values of $z$ (if you are unfamiliar with this fact, just believe it for now; we will prove it in Chapter 4); this contradiction shows that all the coefficients in 2.3 equal 0. Hence $(1,z,\dots,z^m)$ is linearly independent, as claimed.

A list of vectors in $V$ is called ***linearly dependent*** if it is not linearly independent. In other words, a list $(v_1,\dots,v_m)$ of vectors in $V$ is linearly dependent if there exist $a_1,\dots,a_m\in\mathbf{F}$, not all 0, such that $a_1v_1+\cdots+a_mv_m=0$. For example, $\big((2,3,1),(1,-1,2),(7,3,8)\big)$ is linearly dependent in $\mathbf{F}^3$ because

$$2(2,3,1)+3(1,-1,2)+(-1)(7,3,8)=(0,0,0).$$

As another example, any list of vectors containing the 0 vector is linearly dependent (why?).

You should verify that a list $(v)$ of length 1 is linearly independent if and only if $v\neq 0$. You should also verify that a list of length 2 is linearly independent if and only if neither vector is a scalar multiple of the other. Caution: a list of length three or more may be linearly dependent even though no vector in the list is a scalar multiple of any other vector in the list, as shown by the example in the previous paragraph.

If some vectors are removed from a linearly independent list, the remaining list is also linearly independent, as you should verify. To allow this to remain true even if we remove all the vectors, we declare the empty list $()$ to be linearly independent.

The lemma below will often be useful. It states that given a linearly dependent list of vectors, with the first vector not zero, one of the vectors is in the span of the previous ones and furthermore we can throw out that vector without changing the span of the original list.

**2.4 Linear Dependence Lemma:** *If $(v_1, \ldots, v_m)$ is linearly dependent in $V$ and $v_1 \neq 0$, then there exists $j \in \{2, \ldots, m\}$ such that the following hold:*

(a) $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$;

(b) *if the $j^{th}$ term is removed from $(v_1, \ldots, v_m)$, the span of the remaining list equals* $\operatorname{span}(v_1, \ldots, v_m)$.

PROOF: Suppose $(v_1, \ldots, v_m)$ is linearly dependent in $V$ and $v_1 \neq 0$. Then there exist $a_1, \ldots, a_m \in \mathbf{F}$, not all 0, such that

$$a_1 v_1 + \cdots + a_m v_m = 0.$$

Not all of $a_2, a_3, \ldots, a_m$ can be 0 (because $v_1 \neq 0$). Let $j$ be the largest element of $\{2, \ldots, m\}$ such that $a_j \neq 0$. Then

**2.5**
$$v_j = -\frac{a_1}{a_j} v_1 - \cdots - \frac{a_{j-1}}{a_j} v_{j-1},$$

proving (a).

To prove (b), suppose that $u \in \operatorname{span}(v_1, \ldots, v_m)$. Then there exist $c_1, \ldots, c_m \in \mathbf{F}$ such that

$$u = c_1 v_1 + \cdots + c_m v_m.$$

In the equation above, we can replace $v_j$ with the right side of 2.5, which shows that $u$ is in the span of the list obtained by removing the $j^{\text{th}}$ term from $(v_1, \ldots, v_m)$. Thus (b) holds. ■

Now we come to a key result. It says that linearly independent lists are never longer than spanning lists.

**2.6 Theorem:** *In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.*

*Suppose that for each positive integer $m$, there exists a linearly independent list of $m$ vectors in $V$. Then this theorem implies that $V$ is infinite dimensional.*

PROOF: Suppose that $(u_1, \ldots, u_m)$ is linearly independent in $V$ and that $(w_1, \ldots, w_n)$ spans $V$. We need to prove that $m \leq n$. We do so through the multistep process described below; note that in each step we add one of the $u$'s and remove one of the $w$'s.

**Step 1**

The list $(w_1, \dots, w_n)$ spans $V$, and thus adjoining any vector to it produces a linearly dependent list. In particular, the list

$$(u_1, w_1, \dots, w_n)$$

is linearly dependent. Thus by the linear dependence lemma (2.4), we can remove one of the $w$'s so that the list $B$ (of length $n$) consisting of $u_1$ and the remaining $w$'s spans $V$.

**Step j**

The list $B$ (of length $n$) from step $j-1$ spans $V$, and thus adjoining any vector to it produces a linearly dependent list. In particular, the list of length $(n+1)$ obtained by adjoining $u_j$ to $B$, placing it just after $u_1, \dots, u_{j-1}$, is linearly dependent. By the linear dependence lemma (2.4), one of the vectors in this list is in the span of the previous ones, and because $(u_1, \dots, u_j)$ is linearly independent, this vector must be one of the $w$'s, not one of the $u$'s. We can remove that $w$ from $B$ so that the new list $B$ (of length $n$) consisting of $u_1, \dots, u_j$ and the remaining $w$'s spans $V$.

After step $m$, we have added all the $u$'s and the process stops. If at any step we added a $u$ and had no more $w$'s to remove, then we would have a contradiction. Thus there must be at least as many $w$'s as $u$'s. ■

Our intuition tells us that any vector space contained in a finite-dimensional vector space should also be finite dimensional. We now prove that this intuition is correct.

**2.7 Proposition:** *Every subspace of a finite-dimensional vector space is finite dimensional.*

PROOF: Suppose $V$ is finite dimensional and $U$ is a subspace of $V$. We need to prove that $U$ is finite dimensional. We do this through the following multistep construction.

**Step 1**

If $U = \{0\}$, then $U$ is finite dimensional and we are done. If $U \neq \{0\}$, then choose a nonzero vector $v_1 \in U$.

**Step j**

If $U = \operatorname{span}(v_1, \dots, v_{j-1})$, then $U$ is finite dimensional and we are done. If $U \neq \operatorname{span}(v_1, \ldots, v_{j-1})$, then choose a vector $v_j \in U$ such that

$$v_j \notin \operatorname{span}(v_1, \ldots, v_{j-1}).$$

After each step, as long as the process continues, we have constructed a list of vectors such that no vector in this list is in the span of the previous vectors. Thus after each step we have constructed a linearly independent list, by the linear dependence lemma (2.4). This linearly independent list cannot be longer than any spanning list of $V$ (by 2.6), and thus the process must eventually terminate, which means that $U$ is finite dimensional. ∎

## Bases

A ***basis*** of $V$ is a list of vectors in $V$ that is linearly independent and spans $V$. For example,

$$\big((1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\big)$$

is a basis of $\mathbf{F}^n$, called the ***standard basis*** of $\mathbf{F}^n$. In addition to the standard basis, $\mathbf{F}^n$ has many other bases. For example, $\big((1, 2), (3, 5)\big)$ is a basis of $\mathbf{F}^2$. The list $\big((1, 2)\big)$ is linearly independent but is not a basis of $\mathbf{F}^2$ because it does not span $\mathbf{F}^2$. The list $\big((1, 2), (3, 5), (4, 7)\big)$ spans $\mathbf{F}^2$ but is not a basis because it is not linearly independent. As another example, $(1, z, \ldots, z^m)$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

The next proposition helps explain why bases are useful.

**2.8 Proposition:** *A list $(v_1, \ldots, v_n)$ of vectors in $V$ is a basis of $V$ if and only if every $v \in V$ can be written uniquely in the form*

**2.9**
$$v = a_1 v_1 + \cdots + a_n v_n,$$

*where $a_1, \ldots, a_n \in \mathbf{F}$.*

*This proof is essentially a repetition of the ideas that led us to the definition of linear independence.*

Proof: First suppose that $(v_1, \ldots, v_n)$ is a basis of $V$. Let $v \in V$. Because $(v_1, \ldots, v_n)$ spans $V$, there exist $a_1, \ldots, a_n \in \mathbf{F}$ such that 2.9 holds. To show that the representation in 2.9 is unique, suppose that $b_1, \ldots, b_n$ are scalars so that we also have

$$v = b_1 v_1 + \cdots + b_n v_n.$$

Subtracting the last equation from 2.9, we get

$$0 = (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n.$$

This implies that each $a_j - b_j = 0$ (because $(v_1, \dots, v_n)$ is linearly independent) and hence $a_1 = b_1, \dots, a_n = b_n$. We have the desired uniqueness, completing the proof in one direction.

For the other direction, suppose that every $v \in V$ can be written uniquely in the form given by 2.9. Clearly this implies that $(v_1, \dots, v_n)$ spans $V$. To show that $(v_1, \dots, v_n)$ is linearly independent, suppose that $a_1, \dots, a_n \in \mathbf{F}$ are such that

$$0 = a_1 v_1 + \cdots + a_n v_n.$$

The uniqueness of the representation 2.9 (with $v = 0$) implies that $a_1 = \cdots = a_n = 0$. Thus $(v_1, \dots, v_n)$ is linearly independent and hence is a basis of $V$. ■

A spanning list in a vector space may not be a basis because it is not linearly independent. Our next result says that given any spanning list, some of the vectors in it can be discarded so that the remaining list is linearly independent and still spans the vector space.

**2.10 Theorem:** *Every spanning list in a vector space can be reduced to a basis of the vector space.*

PROOF: Suppose $(v_1, \dots, v_n)$ spans $V$. We want to remove some of the vectors from $(v_1, \dots, v_n)$ so that the remaining vectors form a basis of $V$. We do this through the multistep process described below. Start with $B = (v_1, \dots, v_n)$.

**Step 1**

If $v_1 = 0$, delete $v_1$ from $B$. If $v_1 \neq 0$, leave $B$ unchanged.

**Step j**

If $v_j$ is in $\operatorname{span}(v_1, \dots, v_{j-1})$, delete $v_j$ from $B$. If $v_j$ is not in $\operatorname{span}(v_1, \dots, v_{j-1})$, leave $B$ unchanged.

Stop the process after step $n$, getting a list $B$. This list $B$ spans $V$ because our original list spanned $B$ and we have discarded only vectors that were already in the span of the previous vectors. The process insures that no vector in $B$ is in the span of the previous ones. Thus $B$ is linearly independent, by the linear dependence lemma (2.4). Hence $B$ is a basis of $V$. ■

Consider the list

$$((1,2),(3,6),(4,7),(5,9)),$$

which spans $\mathbf{F}^2$. To make sure that you understand the last proof, you should verify that the process in the proof produces $((1,2),(4,7))$, a basis of $\mathbf{F}^2$, when applied to the list above.

Our next result, an easy corollary of the last theorem, tells us that every finite-dimensional vector space has a basis.

**2.11 Corollary:** *Every finite-dimensional vector space has a basis.*

Proof: By definition, a finite-dimensional vector space has a spanning list. The previous theorem tells us that any spanning list can be reduced to a basis. ■

We have crafted our definitions so that the finite-dimensional vector space $\{0\}$ is not a counterexample to the corollary above. In particular, the empty list $()$ is a basis of the vector space $\{0\}$ because this list has been defined to be linearly independent and to have span $\{0\}$.

Our next theorem is in some sense a dual of 2.10, which said that every spanning list can be reduced to a basis. Now we show that given any linearly independent list, we can adjoin some additional vectors so that the extended list is still linearly independent but also spans the space.

**2.12 Theorem:** *Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.*

*This theorem can be used to give another proof of the previous corollary. Specifically, suppose $V$ is finite dimensional. This theorem implies that the empty list $()$ can be extended to a basis of $V$. In particular, $V$ has a basis.*

Proof: Suppose $V$ is finite dimensional and $(v_1,\dots,v_m)$ is linearly independent in $V$. We want to extend $(v_1,\dots,v_m)$ to a basis of $V$. We do this through the multistep process described below. First we let $(w_1,\dots,w_n)$ be any list of vectors in $V$ that spans $V$.

**Step 1**

If $w_1$ is in the span of $(v_1,\dots,v_m)$, let $B=(v_1,\dots,v_m)$. If $w_1$ is not in the span of $(v_1,\dots,v_m)$, let $B=(v_1,\dots,v_m,w_1)$.

**Step j**

If $w_j$ is in the span of $B$, leave $B$ unchanged. If $w_j$ is not in the span of $B$, extend $B$ by adjoining $w_j$ to it.

After each step, $B$ is still linearly independent because otherwise the linear dependence lemma (2.4) would give a contradiction (recall that $(v_1, \dots, v_m)$ is linearly independent and any $w_j$ that is adjoined to $B$ is not in the span of the previous vectors in $B$). After step $n$, the span of $B$ includes all the $w$'s. Thus the $B$ obtained after step $n$ spans $V$ and hence is a basis of $V$. ∎

As a nice application of the theorem above, we now show that every subspace of a finite-dimensional vector space can be paired with another subspace to form a direct sum of the whole space.

*Using the same basic ideas but considerably more advanced tools, this proposition can be proved without the hypothesis that $V$ is finite dimensional.*

**2.13 Proposition:** *Suppose $V$ is finite dimensional and $U$ is a subspace of $V$. Then there is a subspace $W$ of $V$ such that $V = U \oplus W$.*

PROOF: Because $V$ is finite dimensional, so is $U$ (see 2.7). Thus there is a basis $(u_1, \dots, u_m)$ of $U$ (see 2.11). Of course $(u_1, \dots, u_m)$ is a linearly independent list of vectors in $V$, and thus it can be extended to a basis $(u_1, \dots, u_m, w_1, \dots, w_n)$ of $V$ (see 2.12). Let $W = \operatorname{span}(w_1, \dots, w_n)$.

To prove that $V = U \oplus W$, we need to show that

$$V = U + W \quad \text{and} \quad U \cap W = \{0\};$$

see 1.9. To prove the first equation, suppose that $v \in V$. Then, because the list $(u_1, \dots, u_m, w_1, \dots, w_n)$ spans $V$, there exist scalars $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbf{F}$ such that

$$v = \underbrace{a_1 u_1 + \cdots + a_m u_m}_{u} + \underbrace{b_1 w_1 + \cdots + b_n w_n}_{w}.$$

In other words, we have $v = u + w$, where $u \in U$ and $w \in W$ are defined as above. Thus $v \in U + W$, completing the proof that $V = U + W$.

To show that $U \cap W = \{0\}$, suppose $v \in U \cap W$. Then there exist scalars $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbf{F}$ such that

$$v = a_1 u_1 + \cdots + a_m u_m = b_1 w_1 + \cdots + b_n w_n.$$

Thus

$$a_1u_1 + \cdots + a_mu_m - b_1w_1 - \cdots - b_nw_n = 0.$$

Because $(u_1, \ldots, u_m, w_1, \ldots, w_n)$ is linearly independent, this implies that $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$. Thus $v = 0$, completing the proof that $U \cap W = \{0\}$. ■

# Dimension

Though we have been discussing finite-dimensional vector spaces, we have not yet defined the dimension of such an object. How should dimension be defined? A reasonable definition should force the dimension of $\mathbf{F}^n$ to equal $n$. Notice that the basis

$$\big((1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\big)$$

has length $n$. Thus we are tempted to define the dimension as the length of a basis. However, a finite-dimensional vector space in general has many different bases, and our attempted definition makes sense only if all bases in a given vector space have the same length. Fortunately that turns out to be the case, as we now show.

**2.14 Theorem:** *Any two bases of a finite-dimensional vector space have the same length.*

PROOF: Suppose $V$ is finite dimensional. Let $B_1$ and $B_2$ be any two bases of $V$. Then $B_1$ is linearly independent in $V$ and $B_2$ spans $V$, so the length of $B_1$ is at most the length of $B_2$ (by 2.6). Interchanging the roles of $B_1$ and $B_2$, we also see that the length of $B_2$ is at most the length of $B_1$. Thus the length of $B_1$ must equal the length of $B_2$, as desired. ■

Now that we know that any two bases of a finite-dimensional vector space have the same length, we can formally define the dimension of such spaces. The ***dimension*** of a finite-dimensional vector space is defined to be the length of any basis of the vector space. The dimension of $V$ (if $V$ is finite dimensional) is denoted by $\dim V$. As examples, note that $\dim \mathbf{F}^n = n$ and $\dim \mathcal{P}_m(\mathbf{F}) = m + 1$.

Every subspace of a finite-dimensional vector space is finite dimensional (by 2.7) and so has a dimension. The next result gives the expected inequality about the dimension of a subspace.

**2.15 Proposition:** *If $V$ is finite dimensional and $U$ is a subspace of $V$, then* $\dim U \leq \dim V$.

PROOF: Suppose that $V$ is finite dimensional and $U$ is a subspace of $V$. Any basis of $U$ is a linearly independent list of vectors in $V$ and thus can be extended to a basis of $V$ (by 2.12). Hence the length of a basis of $U$ is less than or equal to the length of a basis of $V$. ■

*The real vector space $\mathbf{R}^2$ has dimension 2; the complex vector space $\mathbf{C}$ has dimension 1. As sets, $\mathbf{R}^2$ can be identified with $\mathbf{C}$ (and addition is the same on both spaces, as is scalar multiplication by real numbers). Thus when we talk about the dimension of a vector space, the role played by the choice of $\mathbf{F}$ cannot be neglected.*

To check that a list of vectors in $V$ is a basis of $V$, we must, according to the definition, show that the list in question satisfies two properties: it must be linearly independent and it must span $V$. The next two results show that if the list in question has the right length, then we need only check that it satisfies one of the required two properties. We begin by proving that every spanning list with the right length is a basis.

**2.16 Proposition:** *If $V$ is finite dimensional, then every spanning list of vectors in $V$ with length* $\dim V$ *is a basis of $V$.*

PROOF: Suppose $\dim V = n$ and $(v_1, \ldots, v_n)$ spans $V$. The list $(v_1, \ldots, v_n)$ can be reduced to a basis of $V$ (by 2.10). However, every basis of $V$ has length $n$, so in this case the reduction must be the trivial one, meaning that no elements are deleted from $(v_1, \ldots, v_n)$. In other words, $(v_1, \ldots, v_n)$ is a basis of $V$, as desired. ■

Now we prove that linear independence alone is enough to ensure that a list with the right length is a basis.

**2.17 Proposition:** *If $V$ is finite dimensional, then every linearly independent list of vectors in $V$ with length* $\dim V$ *is a basis of $V$.*

PROOF: Suppose $\dim V = n$ and $(v_1, \ldots, v_n)$ is linearly independent in $V$. The list $(v_1, \ldots, v_n)$ can be extended to a basis of $V$ (by 2.12). However, every basis of $V$ has length $n$, so in this case the extension must be the trivial one, meaning that no elements are adjoined to $(v_1, \ldots, v_n)$. In other words, $(v_1, \ldots, v_n)$ is a basis of $V$, as desired. ■

As an example of how the last proposition can be applied, consider the list $\big((5,7), (4,3)\big)$. This list of two vectors in $\mathbf{F}^2$ is obviously linearly independent (because neither vector is a scalar multiple of the other).

Because $\mathbf{F}^2$ has dimension 2, the last proposition implies that this linearly independent list of length 2 is a basis of $\mathbf{F}^2$ (we do not need to bother checking that it spans $\mathbf{F}^2$).

The next theorem gives a formula for the dimension of the sum of two subspaces of a finite-dimensional vector space.

**2.18 Theorem:** *If $U_1$ and $U_2$ are subspaces of a finite-dimensional vector space, then*

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

*This formula for the dimension of the sum of two subspaces is analogous to a familiar counting formula: the number of elements in the union of two finite sets equals the number of elements in the first set, plus the number of elements in the second set, minus the number of elements in the intersection of the two sets.*

PROOF: Let $(u_1, \ldots, u_m)$ be a basis of $U_1 \cap U_2$; thus $\dim(U_1 \cap U_2) = m$. Because $(u_1, \ldots, u_m)$ is a basis of $U_1 \cap U_2$, it is linearly independent in $U_1$ and hence can be extended to a basis $(u_1, \ldots, u_m, v_1, \ldots, v_j)$ of $U_1$ (by 2.12). Thus $\dim U_1 = m + j$. Also extend $(u_1, \ldots, u_m)$ to a basis $(u_1, \ldots, u_m, w_1, \ldots, w_k)$ of $U_2$; thus $\dim U_2 = m + k$.

We will show that $(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k)$ is a basis of $U_1 + U_2$. This will complete the proof because then we will have

$$\begin{aligned}
\dim(U_1 + U_2) &= m + j + k \\
&= (m + j) + (m + k) - m \\
&= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).
\end{aligned}$$

Clearly $\operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k)$ contains $U_1$ and $U_2$ and hence contains $U_1 + U_2$. So to show that this list is a basis of $U_1 + U_2$ we need only show that it is linearly independent. To prove this, suppose

$$a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_jv_j + c_1w_1 + \cdots + c_kw_k = 0,$$

where all the $a$'s, $b$'s, and $c$'s are scalars. We need to prove that all the $a$'s, $b$'s, and $c$'s equal 0. The equation above can be rewritten as

$$c_1w_1 + \cdots + c_kw_k = -a_1u_1 - \cdots - a_mu_m - b_1v_1 - \cdots - b_jv_j,$$

which shows that $c_1w_1 + \cdots + c_kw_k \in U_1$. All the $w$'s are in $U_2$, so this implies that $c_1w_1 + \cdots + c_kw_k \in U_1 \cap U_2$. Because $(u_1, \ldots, u_m)$ is a basis of $U_1 \cap U_2$, we can write

$$c_1w_1 + \cdots + c_kw_k = d_1u_1 + \cdots + d_mu_m$$

for some choice of scalars $d_1, \dots, d_m$. But $(u_1, \dots, u_m, w_1, \dots, w_k)$ is linearly independent, so the last equation implies that all the $c$'s (and $d$'s) equal 0. Thus our original equation involving the $a$'s, $b$'s, and $c$'s becomes

$$a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_j v_j = 0.$$

This equation implies that all the $a$'s and $b$'s are 0 because the list $(u_1, \dots, u_m, v_1, \dots, v_j)$ is linearly independent. We now know that all the $a$'s, $b$'s, and $c$'s equal 0, as desired. ■

The next proposition shows that dimension meshes well with direct sums. This result will be useful in later chapters.

*Recall that direct sum is analogous to disjoint union. Thus 2.19 is analogous to the statement that if a finite set $B$ is written as $A_1 \cup \cdots \cup A_m$ and the sum of the number of elements in the $A$'s equals the number of elements in $B$, then the union is a disjoint union.*

**2.19 Proposition:** *Suppose $V$ is finite dimensional and $U_1, \dots, U_m$ are subspaces of $V$ such that*

**2.20** $$V = U_1 + \cdots + U_m$$

*and*

**2.21** $$\dim V = \dim U_1 + \cdots + \dim U_m.$$

*Then $V = U_1 \oplus \cdots \oplus U_m$.*

PROOF: Choose a basis for each $U_j$. Put these bases together in one list, forming a list that spans $V$ (by 2.20) and has length $\dim V$ (by 2.21). Thus this list is a basis of $V$ (by 2.16), and in particular it is linearly independent.

Now suppose that $u_1 \in U_1, \dots, u_m \in U_m$ are such that

$$0 = u_1 + \cdots + u_m.$$

We can write each $u_j$ as a linear combination of the basis vectors (chosen above) of $U_j$. Substituting these linear combinations into the expression above, we have written 0 as a linear combination of the basis of $V$ constructed above. Thus all the scalars used in this linear combination must be 0. Thus each $u_j = 0$, which proves that $V = U_1 \oplus \cdots \oplus U_m$ (by 1.8). ■

# Exercises

1. Prove that if $(v_1, \ldots, v_n)$ spans $V$, then so does the list

$$(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$$

obtained by subtracting from each vector (except the last one) the following vector.

2. Prove that if $(v_1, \ldots, v_n)$ is linearly independent in $V$, then so is the list

$$(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$$

obtained by subtracting from each vector (except the last one) the following vector.

3. Suppose $(v_1, \ldots, v_n)$ is linearly independent in $V$ and $w \in V$. Prove that if $(v_1 + w, \ldots, v_n + w)$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_n)$.

4. Suppose $m$ is a positive integer. Is the set consisting of 0 and all polynomials with coefficients in $\mathbf{F}$ and with degree equal to $m$ a subspace of $\mathcal{P}(\mathbf{F})$?

5. Prove that $\mathbf{F}^\infty$ is infinite dimensional.

6. Prove that the real vector space consisting of all continuous real-valued functions on the interval $[0, 1]$ is infinite dimensional.

7. Prove that $V$ is infinite dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in $V$ such that $(v_1, \ldots, v_n)$ is linearly independent for every positive integer $n$.

8. Let $U$ be the subspace of $\mathbf{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of $U$.

9. Prove or disprove: there exists a basis $(p_0, p_1, p_2, p_3)$ of $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials $p_0, p_1, p_2, p_3$ has degree 2.

10. Suppose that $V$ is finite dimensional, with $\dim V = n$. Prove that there exist one-dimensional subspaces $U_1, \ldots, U_n$ of $V$ such that

$$V = U_1 \oplus \cdots \oplus U_n.$$

11. Suppose that $V$ is finite dimensional and $U$ is a subspace of $V$ such that $\dim U = \dim V$. Prove that $U = V$.

12. Suppose that $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_j(2) = 0$ for each $j$. Prove that $(p_0, p_1, \ldots, p_m)$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

13. Suppose $U$ and $W$ are subspaces of $\mathbf{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbf{R}^8$. Prove that $U \cap W = \{0\}$.

14. Suppose that $U$ and $W$ are both five-dimensional subspaces of $\mathbf{R}^9$. Prove that $U \cap W \neq \{0\}$.

15. You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if $U_1, U_2, U_3$ are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(U_1 + U_2 + U_3)&\\ = \dim U_1 &+ \dim U_2 + \dim U_3\\ &- \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3)\\ &+ \dim(U_1 \cap U_2 \cap U_3).\end{aligned}$$

Prove this or give a counterexample.

16. Prove that if $V$ is finite dimensional and $U_1, \ldots, U_m$ are subspaces of $V$, then

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m.$$

17. Suppose $V$ is finite dimensional. Prove that if $U_1, \ldots, U_m$ are subspaces of $V$ such that $V = U_1 \oplus \cdots \oplus U_m$, then

$$\dim V = \dim U_1 + \cdots + \dim U_m.$$

*This exercise deepens the analogy between direct sums of subspaces and disjoint unions of subsets. Specifically, compare this exercise to the following obvious statement: if a finite set is written as a disjoint union of subsets, then the number of elements in the set equals the sum of the number of elements in the disjoint subsets.*