Inner Product Spaces

Branko Ćurgus

March 4, 2015 at 09:05

1 Inner Product Spaces

We will first introduce several "dot-product-like" objects. We start with the most general.

Definition 1.1. Let $\mathscr{V}$ be a vector space over a scalar field $\mathbb{F}$. A function

$$[\,\cdot\,,\cdot\,] : \mathscr{V} \times \mathscr{V} \to \mathbb{F}$$

is a *sesquilinear form* on $\mathscr{V}$ if the following two conditions are satisfied.

(a) (linearity in the first variable) $\forall\, \alpha, \beta \in \mathbb{F} \quad \forall\, u, v, w \in \mathscr{V}$

$$[\alpha u + \beta v, w] = \alpha[u, w] + \beta[v, w].$$

(b) (anti-linearity in the second variable) $\forall\, \alpha, \beta \in \mathbb{F} \quad \forall\, u, v, w \in \mathscr{V} \quad [u, \alpha v + \beta w] = \overline{\alpha}[u, v] + \overline{\beta}[u, w]$.

Example 1.2. Let $M \in \mathbb{C}^{n\times n}$ be arbitrary. Then

$$[\mathbf{x}, \mathbf{y}] = (M\mathbf{x}) \cdot \mathbf{y}, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{C}^n,$$

is a sesquilinear form on the complex vector space $\mathbb{C}^n$. Here $\cdot$ denotes the usual dot product in $\mathbb{C}$.

An abstract form of the Pythagorean Theorem holds for sesquilinear forms.

Theorem 1.3 (Pythagorean Theorem)**.** *Let $[\,\cdot\,,\cdot\,]$ be a sesquilinear form on a vector space $\mathscr{V}$ over a scalar field $\mathbb{F}$. If $v_1, \cdots, v_n \in \mathscr{V}$ are such that $[v_j, v_k] = 0$ whenever $j \neq k, j, k \in \{1, \ldots, n\}$, then*

$$\left[\sum_{j=1}^{n} v_j, \sum_{k=1}^{n} v_k\right] = \sum_{j=1}^{n} [v_j, v_j].$$

Proof. Assume that $[v_j, v_k] = 0$ whenever $j \neq k, j, k \in \{1, \ldots, n\}$ and apply the additivity of the sesquilinear form in both variables to get:

$$\begin{aligned}\left[\sum_{j=1}^{n} v_j, \sum_{k=1}^{n} v_k\right] &= \sum_{j=1}^{n}\sum_{k=1}^{n}[v_j, v_k] \\ &= \sum_{j=1}^{n}[v_j, v_j]. \qquad \square\end{aligned}$$

Theorem 1.4 (Polarization identity)**.** *Let $\mathscr{V}$ be a vector space over a scalar field $\mathbb{F}$ and let $[\,\cdot\,,\cdot\,] : \mathscr{V} \times \mathscr{V} \to \mathbb{F}$ be a sesquilinear form on $\mathscr{V}$. If* $\mathrm{i} \in \mathbb{F}$, *then*

$$[u, v] = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \big[u + \mathrm{i}^k v, u + \mathrm{i}^k v\big] \tag{1}$$

for all $u, v \in \mathscr{V}$.

Corollary 1.5. *Let $\mathscr{V}$ be a vector space over a scalar field $\mathbb{F}$ and let $[\,\cdot\,,\cdot\,] : \mathscr{V} \times \mathscr{V} \to \mathbb{F}$ be a sesquilinear form on $\mathscr{V}$. If* $\mathrm{i} \in \mathbb{F}$ *and* $[v, v] = 0$ *for all $v \in \mathscr{V}$, then* $[u, v] = 0$ *for all $u, v \in \mathscr{V}$.*

Definition 1.6. Let $\mathscr{V}$ be a vector space over a scalar field $\mathbb{F}$. A sesquilinear form $[\,\cdot\,,\cdot\,] : \mathscr{V} \times \mathscr{V} \to \mathbb{F}$ is *hermitian* if

(c) (hermiticity) $\quad \forall\, u, v \in \mathscr{V} \quad \overline{[u, v]} = [v, u]$.

A hermitian sesquilinear form is also called an *inner product.*

Corollary 1.7. *Let $\mathscr{V}$ be a vector space over a scalar field $\mathbb{F}$ such that* $\mathrm{i} \in \mathbb{F}$. *Let $[\,\cdot\,,\cdot\,] : \mathscr{V} \times \mathscr{V} \to \mathbb{F}$ be a sesquilinear form on $\mathscr{V}$. Then $[\,\cdot\,,\cdot\,]$ is hermitian if and only if* $[v, v] \in \mathbb{R}$ *for all $v \in \mathscr{V}$.*

Proof. The "only if" direction follows from the definition of a hermitian sesquilinear form. To prove "if" direction assume that $[v, v] \in \mathbb{R}$ for all $v \in \mathscr{V}$. Let $u, v \in \mathscr{V}$ be arbitrary. By assumption $\big[u + \mathrm{i}^k v, u + \mathrm{i}^k v\big] \in \mathbb{R}$ for all $k \in \{0, 1, 2, 3\}$. Therefore

$$\begin{aligned}\overline{[u, v]} &= \frac{1}{4}\sum_{k=0}^{3}(-\mathrm{i})^k \big[u + \mathrm{i}^k v, u + \mathrm{i}^k v\big] \\ &= \frac{1}{4}\sum_{k=0}^{3}(-\mathrm{i})^k \mathrm{i}^k (-\mathrm{i})^k \big[(-\mathrm{i})^k u + v, (-\mathrm{i})^k u + v\big]\end{aligned}$$

$$= \frac{1}{4}\sum_{k=0}^{3}(-\mathrm{i})^k \big[v + (-\mathrm{i})^k u, v + (-\mathrm{i})^k u\big].$$

Notice that the values of $(-\mathrm{i})^k$ at $k = 0, 1, 2, 3$, in this particular order are: $1, -\mathrm{i}, -1, \mathrm{i}$. These are exactly the values of i^k in the order $k = 0, 3, 2, 1$. Therefore rearranging the order of terms in the last four-term-sum we have

$$\frac{1}{4}\sum_{k=0}^{3}(-\mathrm{i})^k \big[v + (-\mathrm{i})^k u, v + (-\mathrm{i})^k u\big] = \frac{1}{4}\sum_{k=0}^{3}\mathrm{i}^k \big[v + \mathrm{i}^k u, v + \mathrm{i}^k u\big].$$

Together with Theorem 1.4, the last two displayed equalities yield $\overline{[u, v]} = [v, u]$. □

Let $[\,\cdot\,,\cdot\,]$ be an inner product on $\mathscr{V}$. The hermiticity of $[\,\cdot\,,\cdot\,]$ implies that $\overline{[v, v]} = [v, v]$ for all $v \in \mathscr{V}$. Thus $[v, v] \in \mathbb{R}$ for all $v \in \mathscr{V}$. The natural trichotomy that arises is the motivation for the following definition.

Definition 1.8. An inner product $[\,\cdot\,,\cdot\,]$ on $\mathscr{V}$ is called *nonnegative* if $[v, v] \geq 0$ for all $v \in \mathscr{V}$, it is called *nonpositive* if $[v, v] \leq 0$ for all $v \in \mathscr{V}$, and it is called *indefinite* if there exist $u \in \mathscr{V}$ and $v \in \mathscr{V}$ such that $[u, u] < 0$ and $[v, v] > 0$.

2 Nonnegative inner products

The following implication that you might have learned in high school will be useful below.

Theorem 2.1 (High School Theorem)**.** *Let a, b, c be real numbers. Assume $a \geq 0$. Then the following implication holds:*

$$\forall\, x \in \mathbb{Q} \quad ax^2 + bx + c \geq 0 \qquad \Rightarrow \qquad b^2 - 4ac \leq 0. \tag{2}$$

Theorem 2.2 (Cauchy-Bunyakovsky-Schwartz Inequality)**.** *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a nonnegative inner product on $\mathscr{V}$. Then*

$$\forall\, u, v \in \mathscr{V} \quad |\langle u, v\rangle|^2 \leq \langle u, u\rangle\langle v, v\rangle. \tag{3}$$

The equality occurs in (3) *if and only if there exists $\alpha, \beta \in \mathbb{F}$ not both* 0 *such that $\langle \alpha u + \beta v, \alpha u + \beta v\rangle = 0$.*

Proof. Let $u, v \in \mathscr{V}$ be arbitrary. Since $\langle \cdot, \cdot \rangle$ is nonnegative we have

$$\forall\, t \in \mathbb{Q} \qquad \big\langle u + t\langle u, v\rangle v, u + t\langle u, v\rangle v \big\rangle \geq 0. \tag{4}$$

Since $\langle \cdot, \cdot \rangle$ is a sesquilinear hermitian form on $\mathscr{V}$, (4) is equivalent to

$$\forall\, t \in \mathbb{Q} \qquad \langle u, u\rangle + 2t|\langle u, v\rangle|^2 + t^2|\langle u, v\rangle|^2\langle v, v\rangle \geq 0. \tag{5}$$

As $\langle v, v\rangle \geq 0$, the High School Theorem applies and (5) implies

$$4|\langle u, v\rangle|^4 - 4|\langle u, v\rangle|^2\langle u, u\rangle\langle v, v\rangle \leq 0. \tag{6}$$

Again, since $\langle u, u\rangle \geq 0$ and $\langle v, v\rangle \geq 0$, (6) is equivalent to

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle\langle v, v\rangle.$$

Since $u, v \in \mathscr{V}$ were arbitrary, (3) is proved.

Next we prove the claim related to the equality in (3). We first prove the "if" part. Assume that $u, v \in \mathscr{V}$ and $\alpha, \beta \in \mathbb{F}$ are such that $|\alpha|^2 + |\beta|^2 > 0$ and

$$\langle \alpha u + \beta v, \alpha u + \beta v\rangle = 0$$

We need to prove that $|\langle u, v\rangle|^2 = \langle u, u\rangle\langle v, v\rangle$.

Since $|\alpha|^2 + |\beta|^2 > 0$, we have two cases $\alpha \neq 0$ or $\beta \neq 0$. We consider the case $\alpha \neq 0$. The case $\beta \neq 0$ is similar. Set $w = \alpha u + \beta v$. Then $\langle w, w\rangle = 0$ and $u = \gamma v + \delta w$ where $\gamma = -\beta/\alpha$ and $\delta = 1/\alpha$. Notice that the Cauchy-Bunyakovsky-Schwarz inequality and $\langle w, w\rangle = 0$ imply that $\langle w, x\rangle = 0$ for all $x \in \mathscr{V}$. Now we calculate

$$|\langle u, v\rangle| = |\langle \gamma v + \delta w, v\rangle| = |\gamma\langle v, v\rangle + \delta\langle w, v\rangle| = |\gamma\langle v, v\rangle| = |\gamma|\langle v, v\rangle$$

and

$$\langle u, u\rangle = \langle \gamma v + \delta w, \gamma v + \delta w\rangle = \langle \gamma v, \gamma v\rangle = |\gamma|^2\langle v, v\rangle.$$

Thus,

$$|\langle u, v\rangle|^2 = |\gamma|^2\langle v, v\rangle^2 = \langle u, u\rangle\langle v, v\rangle.$$

This completes the proof of the "if" part.

To prove the "only if" part, assume $|\langle u, v\rangle|^2 = \langle u, u\rangle\langle v, v\rangle$. If $\langle v, v\rangle = 0$, then with $\alpha = 0$ and $\beta = 1$ we have

$$\langle \alpha u + \beta v, \alpha u + \beta v\rangle = \langle v, v\rangle = 0.$$

If $\langle v, v\rangle \neq 0$, then with $\alpha = \langle v, v\rangle$ and $\beta = -\langle u, v\rangle$ we have $|\alpha|^2 + |\beta|^2 > 0$ and

$$\langle \alpha u + \beta v, \alpha u + \beta v\rangle = \langle v, v\rangle\bigl(\langle v, v\rangle\langle u, u\rangle - |\langle u, v\rangle|^2 - |\langle u, v\rangle|^2 + |\langle u, v\rangle|^2\bigr) = 0.$$

This completes the proof of the characterization of equality in the Cauchy-Bunyakovsky-Schwartz Inequality. □

Corollary 2.3. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a nonnegative inner product on $\mathscr{V}$. Then the following two implications are equivalent.*

(i) *If $v \in \mathscr{V}$ and $\langle u, v\rangle = 0$ for all $u \in \mathscr{V}$, then $v = 0$.*
(ii) *If $v \in \mathscr{V}$ and $\langle v, v\rangle = 0$, then $v = 0$.*

Proof. Assume that the implication (i) holds and let $v \in \mathscr{V}$ be such that $\langle v, v\rangle = 0$. Let $u \in \mathscr{V}$ be arbitrary. By the the CBS inequality

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle\langle v, v\rangle = 0.$$

Thus, $\langle u, v\rangle = 0$ for all $u \in \mathscr{V}$. By (i) we conclude $v = 0$. This proves (ii).

The converse is trivial. However, here is a proof. Assume that the implication (ii) holds. To prove (i), let $v \in \mathscr{V}$ and assume $\langle u, v\rangle = 0$ for all $u \in \mathscr{V}$. Setting $u = v$ we get $\langle v, v\rangle = 0$. Now (ii) yields $v = 0$. □

Definition 2.4. Let $\mathscr{V}$ be a vector space over a scalar field $\mathbb{F}$. An inner product $[\,\cdot\,,\cdot\,]$ on $\mathscr{V}$ is *nondegenerate* if the following implication holds

(d) (nondegenerecy) $\quad u \in \mathscr{V}$ and $[u, v] = 0$ for all $v \in \mathscr{V}$ implies $u = 0$.

We conclude this section with a characterization of the best approximation property.

Theorem 2.5 (Best Approximation-Orthogonality Theorem)**.** *Let $(\mathscr{V}, \langle\cdot,\cdot\rangle)$ be an inner product space with a nonnegative inner product. Let $\mathscr{U}$ be a subspace of $\mathscr{V}$. Let $v \in \mathscr{V}$ and $u_0 \in \mathscr{U}$. Then*

$$\forall\, u \in \mathscr{U} \qquad \langle v - u_0, v - u_0\rangle \leq \langle v - u, v - u\rangle. \tag{7}$$

if and only if

$$\forall\, u \in \mathscr{U} \qquad \langle v - u_0, u\rangle = 0. \tag{8}$$

Proof. First we prove the "only if" part. Assume (7). Let $u \in \mathscr{U}$ be arbitrary. Set $\alpha = \langle v - u_0, u\rangle$. Clearly $\alpha \in \mathbb{F}$. Let $t \in \mathbb{Q} \subseteq \mathbb{F}$ be arbitrary. Since $u_0 - t\alpha u \in \mathscr{U}$, (7) implies

$$\forall\, t \in \mathbb{Q} \qquad \langle v - u_0, v - u_0\rangle \leq \langle v - u_0 + t\alpha u, v - u_0 + t\alpha u\rangle. \tag{9}$$

Now recall that $\alpha = \langle v - u_0, u\rangle$ and expand the right-hand side of (9):

$$\begin{aligned}\langle v - u_0 + t\alpha u, v - u_0 + t\alpha u\rangle &= \langle v - u_0, v - u_0\rangle + \langle v - u_0, t\alpha u\rangle \\ &\qquad + \langle t\alpha u, v - u_0\rangle + \langle t\alpha u, t\alpha u\rangle \\ &= \langle v - u_0, v - u_0\rangle + t\overline{\alpha}\langle v - u_0, u\rangle \\ &\qquad + t\alpha\langle u, v - u_0\rangle + t^2|\alpha|^2\langle u, u\rangle \\ &= \langle v - u_0, v - u_0\rangle + 2t|\alpha|^2 + t^2|\alpha|^2\langle u, u\rangle.\end{aligned}$$

Thus (9) is equivalent to

$$\forall\, t \in \mathbb{Q} \qquad 0 \leq 2t|\alpha|^2 + t^2|\alpha|^2\langle u, u\rangle. \tag{10}$$

By the High School Theorem, (10) implies

$$4|\alpha|^4 - 4|\alpha|^2\langle u, u\rangle\, 0 = 4|\alpha|^4 \leq 0.$$

Consequently $\alpha = \langle v - u_0, u\rangle = 0$. Since $u \in \mathscr{U}$ was arbitrary, (8) is proved.

For the "if" part assume that (8) is true. Let $u \in \mathscr{U}$ be arbitrary. Notice that $u_0 - u \in \mathscr{U}$ and calculate

$$\begin{aligned}\langle v - u, v - u\rangle &= \langle v - u_0 + u_0 - u, v - u_0 + u_0 - u\rangle \\ \boxed{\text{by (8) and Pythag. thm.}}\ &= \langle v - u_0 +, v - u_0\rangle + \langle u_0 - u, u_0 - u\rangle \\ \boxed{\text{since } \langle u_0 - u, u_0 - u\rangle \geq 0}\ &\geq \langle v - u_0 +, v - u_0\rangle.\end{aligned}$$

This proves (7). $\square$

3 Positive definite inner products

It follows from Corollary 2.3 that a nonnegative inner product $\langle\,\cdot\,,\cdot\,\rangle$ on $\mathscr{V}$ is nondegenerate if and only if $\langle v, v\rangle = 0$ implies $v = 0$. A nonnegative nondegenerate inner product is also called *positive definite inner product.* Since this is the most often encountered inner product we give its definition as it commonly given in textbooks.

Definition 3.1. Let $\mathscr{V}$ be a vector space over a scalar field $\mathbb{F}$. A function $\langle \cdot , \cdot \rangle : \mathscr{V} \times \mathscr{V} \to \mathbb{F}$ is called a *positive definite inner product* on $\mathscr{V}$ if the following conditions are satisfied;

(a) $\forall\, u, v, w \in \mathscr{V} \ \ \forall\, \alpha, \beta \in \mathbb{F} \quad \langle \alpha u + \beta v, v\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$,

(b) $\forall\, u, v \in \mathscr{V} \quad \langle u, v\rangle = \overline{\langle v, u\rangle}$,

(c) $\forall\, v \in \mathscr{V} \quad \langle v, v\rangle \geq 0$,

(d) If $v \in \mathscr{V}$ and $\langle v, v\rangle = 0$, then $v = 0$.

A positive definite inner product gives rise to a norm.

Theorem 3.2. *Let $(\mathscr{V}, \langle \cdot , \cdot \rangle)$ be a vector space over $\mathbb{F}$ with a positive definite inner product $\langle \cdot , \cdot \rangle$. The function $\| \cdot \| : \mathscr{V} \to \mathbb{R}$ defined by*

$$\|v\| = \sqrt{\langle v, v\rangle}, \quad v \in \mathscr{V},$$

is a norm on $\mathscr{V}$. That is for all $u, v \in \mathscr{V}$ and all $\alpha \in \mathbb{F}$ we have $\|v\| \geq 0$, $\|\alpha v\| = |\alpha|\|v\|$, $\|u + v\| \leq \|u\| + \|v\|$ and $\|v\| = 0$ implies $v = 0_{\mathscr{V}}$.

Definition 3.3. Let $(\mathscr{V}, \langle \cdot , \cdot \rangle)$ be a vector space over $\mathbb{F}$ with a positive definite inner product $\langle \cdot , \cdot \rangle$. A set of vectors $\mathscr{A} \subset \mathscr{V}$ is said to form an *orthogonal system* in $\mathscr{V}$ if for all $u, v \in \mathscr{A}$ we have $\langle u, v\rangle = 0$ whenever $u \neq v$ and for all $v \in \mathscr{A}$ we have $\langle v, v\rangle > 0$. An orthogonal system $\mathscr{A}$ is called an *orthonormal system* if for all $v \in \mathscr{A}$ we have $\langle v, v\rangle = 1$.

Proposition 3.4. *Let $(\mathscr{V}, \langle \cdot , \cdot \rangle)$ be a vector space over $\mathbb{F}$ with a positive definite inner product $\langle \cdot , \cdot \rangle$. Let $u_1, \ldots, u_n$ be an orthogonal system in $\mathscr{V}$. If $v = \sum_{j=1}^{n} \alpha_j u_j$, then $\alpha_j = \langle v, u_j\rangle / \langle u_j, u_j\rangle$. In particular, an orthogonal system is linearly independent.*

Theorem 3.5 (The Gram-Schmidt orthogonalization)**.** *Let $(\mathscr{V}, \langle \cdot , \cdot \rangle)$ be a vector space over $\mathbb{F}$ with a positive definite inner product $\langle \cdot , \cdot \rangle$. Let $n \in \mathbb{N}$ and let $v_1, \ldots, v_n$ be linearly independent vectors in $\mathscr{V}$. Let the vectors $u_1, \ldots, u_n$ be defined recursively by*

$$\begin{aligned} u_1 &= v_1, \\ u_{k+1} &= v_{k+1} - \sum_{j=1}^{k} \frac{\langle v_{k+1}, u_j\rangle}{\langle u_j, u_j\rangle} u_j, \quad k \in \{1, \ldots, n-1\}. \end{aligned}$$

Then the vectors $u_1, \ldots, u_n$ form an orthogonal system which has the same fan as the given vectors $v_1, \ldots, v_n$.

Proof. We will prove by Mathematical Induction the following statement: For all $k \in \{1, \ldots, n\}$ we have:

(a) $\langle u_k, u_k \rangle > 0$ and $\langle u_j, u_k \rangle = 0$ whenever $j \in \{1, \ldots, k-1\}$;
(b) vectors $u_1, \ldots, u_k$ are linearly independent;
(c) $\operatorname{span}\{u_1, \ldots, u_k\} = \operatorname{span}\{v_1, \ldots, v_k\}$.

For $k = 1$ statements (a), (b) and (c) are clearly true. Let $m \in \{1, \ldots, n-1\}$ and assume that statements (a), (b) and (c) are true for all $k \in \{1, \ldots, m\}$.

Next we will prove that statements (a), (b) and (c) are true for $k = m+1$. Recall the definition of u_{m+1}:

$$u_{m+1} = v_{m+1} - \sum_{j=1}^{m} \frac{\langle v_{m+1}, u_j \rangle}{\langle u_j, u_j \rangle} u_j.$$

By the Inductive Hypothesis we have $\operatorname{span}\{u_1, \ldots, u_m\} = \operatorname{span}\{v_1, \ldots, v_m\}$. Since $v_1 \ldots, v_{m+1}$ are linearly independent, $v_{m+1} \notin \operatorname{span}\{u_1, \ldots, u_m\}$. Therefore, $n_{m+1} \neq 0_{\mathscr{V}}$. That is $\langle u_{m+1}, u_{m+1} \rangle > 0$. Let $k \in \{1, \ldots, m\}$ be arbitrary. Then by the Inductive Hypothesis we have that $\langle u_j, u_k \rangle = 0$ whenever $j \in \{1, \ldots, m\}$ and $j \neq k$. Therefore,

$$\begin{aligned}\langle u_{m+1}, u_k \rangle &= \langle v_{m+1}, u_k \rangle - \sum_{j=1}^{m} \frac{\langle v_{m+1}, u_j \rangle}{\langle u_j, u_j \rangle} \langle u_j, u_k \rangle \\ &= \langle v_{m+1}, u_k \rangle - \langle v_{m+1}, u_k \rangle \\ &= 0.\end{aligned}$$

This proves claim (a). To prove claim (b) notice that by the Inductive Hypothesis $u_1, \ldots, u_m$ are linearly independent and $u_{m+1} \notin \operatorname{span}\{u_1, \ldots, u_m\}$ since $v_{m+1} \notin \operatorname{span}\{u_1, \ldots, u_m\}$. To prove claim (c) notice that the definition of u_{m+1} implies $u_{m+1} \in \operatorname{span}\{v_1, \ldots, v_{m+1}\}$. Since by the inductive hypothesis $\operatorname{span}\{u_1, \ldots, u_m\} = \operatorname{span}\{v_1, \ldots, v_m\}$, we have $\operatorname{span}\{u_1, \ldots, u_{m+1}\} \subseteq \operatorname{span}\{v_1, \ldots, v_{m+1}\}$. The converse inclusion follows from the fact that $v_{m+1} \in \operatorname{span}\{u_1, \ldots, u_{m+1}\}$.

It is clear that the claim of the theorem follows from the claim that has been proven. $\square$

The following two statements are immediate consequences of the Gram-Schmidt orthogonalization process.

Corollary 3.6. *If $\mathscr{V}$ is a finite dimensional vector space with positive definite inner product $\langle \cdot, \cdot \rangle$, then $\mathscr{V}$ has an orthonormal basis.*

Corollary 3.7. *If $\mathscr{V}$ is a complex vector space with positive definite inner product and $T \in \mathscr{L}(\mathscr{V})$ then there exists an orthonormal basis $\mathscr{B}$ such that $\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T)$ is upper-triangular.*

Definition 3.8. Let $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ be a positive definite inner product space and $\mathscr{A} \subset \mathscr{V}$. We define $\mathscr{A}^{\perp} = \{v \in \mathscr{V} : \langle v, a \rangle = 0 \; \forall \, a \in \mathscr{A}\}$.

The following is a simple proposition.

Proposition 3.9. *Let $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ be a positive definite inner product space and $\mathscr{A} \subset \mathscr{V}$. Then $A^{\perp}$ is a subspace of $\mathscr{V}$.*

Theorem 3.10. *Let $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ be a positive definite inner product space and let $\mathscr{U}$ be a finite dimensional subspace of $\mathscr{V}$. Then $\mathscr{V} = \mathscr{U} \oplus \mathscr{U}^{\perp}$.*

Proof. We first prove that $\mathscr{V} = \mathscr{U} \oplus \mathscr{U}^{\perp}$. Note that since $\mathscr{U}$ is a subspace of $\mathscr{V}$, $\mathscr{U}$ inherits the positive definite inner product from $\mathscr{V}$. Thus $\mathscr{U}$ is a finite dimensional positive definite inner product space. Thus there exists an orthonormal basis of $\mathscr{U}$, $\mathscr{B} = \{u_1, u_2, \ldots u_k\}$.

Let $v \in \mathscr{V}$ be arbitrary. Then

$$v = \left(\sum_{j=1}^{k} \langle v, u_j \rangle u_j \right) + \left(v - \sum_{j=1}^{k} \langle v, u_j \rangle u_j \right),$$

where the first summand is in $\mathscr{U}$. We will prove that the second summand is in $\mathscr{U}^{\perp}$. Set $w = \sum_{j=1}^{k} \langle v, u_j \rangle u_j \in \mathscr{U}$. We claim that $v - w \in \mathscr{U}^{\perp}$. To prove this claim let $u \in \mathscr{U}$ be arbitrary. Since $\mathscr{B}$ is an orhonormal basis of $\mathscr{U}$, by Proposition 3.4 we have

$$u = \sum_{j=1}^{k} \langle u, u_j \rangle u_j.$$

Therefore

$$\begin{aligned} \langle v - w, u \rangle &= \langle v, u \rangle - \sum_{j=1}^{k} \langle v, u_j \rangle \langle u_j, u \rangle \\ &= \langle v, u \rangle - \left\langle v, \sum_{j=1}^{k} \langle u, u_j \rangle u_j \right\rangle \end{aligned}$$

$$= \langle v, u\rangle - \langle v, u\rangle$$
$$= 0.$$

Thus $\langle v - w, u\rangle = 0$ for all $u \in \mathscr{U}$. That is $v - w \in \mathscr{U}^{\perp}$. This proves that $\mathscr{V} = \mathscr{U} \oplus \mathscr{U}^{\perp}$.

To prove that the sum is direct, let $v \in \mathscr{U}$ and $v \in \mathscr{U}^{\perp}$. Then $\langle v, v\rangle = 0$. Since $\langle\,\cdot\,,\cdot\,\rangle$ is positive definite, this implies $v = 0_{\mathscr{V}}$. The theorem is proved. □

Corollary 3.11. *Let $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle)$ be a positive definite inner product space and let $\mathscr{U}$ be a finite dimensional subspace of $\mathscr{V}$. Then $\bigl(\mathscr{U}^{\perp}\bigr)^{\perp} = \mathscr{U}$.*

Exercise 3.12. Let $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle)$ be a positive definite inner product space and let $\mathscr{U}$ be a subspace of $\mathscr{V}$. Prove that $\Bigl(\bigl(\mathscr{U}^{\perp}\bigr)^{\perp}\Bigr)^{\perp} = \mathscr{U}^{\perp}$.

Recall that an arbitrary direct sum $\mathscr{V} = \mathscr{U} \oplus \mathscr{W}$ gives rise to a projection operator $P_{\mathscr{U}\|\mathscr{W}}$, the projection of $\mathscr{V}$ onto $\mathscr{U}$ parallel to $\mathscr{W}$.

If $\mathscr{V} = \mathscr{U} \oplus \mathscr{U}^{\perp}$, then the resulting projection of $\mathscr{V}$ onto $\mathscr{U}$ parallel to $\mathscr{U}^{\perp}$ is called the *orthogonal projection* of $\mathscr{V}$ onto $\mathscr{U}$; it is denoted simply by $P_{\mathscr{U}}$. By definition for every $v \in \mathscr{V}$,

$$u = P_{\mathscr{U}} v \qquad \Leftrightarrow \qquad u \in \mathscr{U} \quad \text{and} \quad v - u \in \mathscr{U}^{\perp}.$$

As for any projection we have $P_{\mathscr{U}} \in \mathscr{L}(\mathscr{V})$, $\operatorname{ran} P_{\mathscr{U}} = \mathscr{U}$, $\operatorname{nul} P_{\mathscr{U}} = \mathscr{U}^{\perp}$, and $(P_{\mathscr{U}})^2 = P_{\mathscr{U}}$.

Theorems 3.10 and 2.5 yield the following solution of the best approximation problem for finite dimensional subspaces of a vector space with a positive definite inner product.

Corollary 3.13. *Let $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle)$ be a vector space with a positive definite inner product and let $\mathscr{U}$ be a finite dimensional subspace of $\mathscr{V}$. For arbitrary $v \in \mathscr{V}$ the vector $P_{\mathscr{U}} v \in \mathscr{U}$ is the unique best approximation for v in $\mathscr{U}$. That is*

$$\bigl\|v - P_{\mathscr{U}} v\bigr\| \leq \|v - u\| \quad \textit{for all} \quad u \in \mathscr{U}.$$

4 The definition of an adjoint operator

Let $\mathscr{V}$ be a vector space over $\mathbb{F}$. The space $\mathscr{L}(\mathscr{V}, \mathbb{F})$ is called the *dual space* of $\mathscr{V}$; it is denoted by $\mathscr{V}^*$.

Theorem 4.1. *Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$. Define the mapping*

$$\Phi : \mathscr{V} \to \mathscr{V}^*$$

as follows: for $w \in \mathscr{V}$ we set

$$\big(\Phi(w)\big)(v) = \langle v, w\rangle \quad \textit{for all} \quad v \in \mathscr{V}.$$

Then Φ is a anti-linear bijection.

Proof. Clearly, for each $w \in \mathscr{V}$, $\Phi(w) \in \mathscr{V}^*$. The mapping Φ is anti-linear, since for $\alpha, \beta \in \mathbb{F}$ and $u, w \in \mathscr{V}$, for all $v \in \mathscr{V}$ we have

$$\begin{aligned}
\big(\Phi(\alpha u + \beta w)\big)(v) &= \langle v, \alpha u + \beta w\rangle \\
&= \overline{\alpha}\langle v, u\rangle + \overline{\beta}\langle v, a \\
&= \overline{\alpha}\big(\Phi(u)\big)(v) + \overline{\beta}\big(\Phi(w)\big)(v) \\
&= \big(\overline{\alpha}\Phi(u) + \overline{\beta}\Phi(w)\big)(v).
\end{aligned}$$

Thus $\Phi(\alpha u + \beta w) = \overline{\alpha}\Phi(u) + \overline{\beta}\Phi(w)$. This proves anti-linearity.

To prove injectivity of Φ, let $u, w \in \mathscr{V}$ be such that $\Phi(u) = \Phi(w)$. Then $\big(\Phi(u)\big)(v) = \big(\Phi(w)\big)(v)$ for all $v \in \mathscr{V}$. By the definition of Φ this means $\langle v, u\rangle = \langle v, w\rangle$ for all $v \in \mathscr{V}$. Consequently, $\langle v, u - w\rangle = 0$ for all $v \in \mathscr{V}$. In particular, with $v = u - w$ we have $\langle u - w, u - w\rangle = 0$. Since $\langle\,\cdot\,,\cdot\,\rangle$ is a positive definite inner product, it follows that $u - w = 0_{\mathscr{V}}$, that is $u = w$.

To prove that Φ is a surjection we use the assumption that $\mathscr{V}$ is finite dimensional. Then there exists an orthonormal basis $u_1, \ldots, u_n$ of $\mathscr{V}$. Let $\varphi \in \mathscr{V}^*$ be arbitrary. Set

$$w = \sum_{j=1}^{n} \overline{\varphi(u_j)}\, u_j.$$

The proof that $\Phi(w) = \varphi$ follows. Let $v \in \mathscr{V}$ be arbitrary.

$$\begin{aligned}
\big(\Phi(w)\big)(v) &= \langle v, w\rangle \\
&= \left\langle v, \sum_{j=1}^{n} \overline{\varphi(u_j)}\, u_j \right\rangle \\
&= \sum_{j=1}^{n} \varphi(u_j)\langle v, u_j\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \langle v, u_j \rangle \varphi(u_j) \\
&= \varphi\left(\sum_{j=1}^{n} \langle v, u_j \rangle u_j \right) \\
&= \varphi(v).
\end{aligned}
$$

The theorem is proved. □

The mapping Φ from the previous theorem is convenient to define the adjoint of a linear operator. In the next definition we will deal with two positive definite inner product spaces. To emphasize the different inner products and different mappings Φ we will use subscripts.

Let $\big(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\big)$ and $\big(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\big)$ be two finite dimensional vector spaces over the same scalar field $\mathbb{F}$ and with positive definite inner products. Let $T \in \mathscr{L}(\mathscr{V}, \mathscr{W})$. We define the adjoint $T^* : \mathscr{W} \to \mathscr{V}$ of T by

$$
T^* w = \Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(w) \circ T\big), \qquad w \in \mathscr{W}. \tag{11}
$$

Since $\Phi_{\mathscr{W}}$ and $\Phi_{\mathscr{V}}^{-1}$ are anti-linear, T^* is linear For arbitrary $\alpha_1, \alpha_1 \in \mathbb{F}$ and $w_1, w_2 \in \mathscr{V}$ we have

$$
\begin{aligned}
T^*(\alpha_1 w_1 + \alpha_2 w_2) &= \Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(\alpha_1 w_1 + \alpha_2 w_2) \circ T\big) \\
&= \Phi_{\mathscr{V}}^{-1}\big(\big(\overline{\alpha}_1 \Phi_{\mathscr{W}}(w_1) + \overline{\alpha}_2 \Phi_{\mathscr{W}}(w_2)\big) \circ T\big) \\
&= \Phi_{\mathscr{V}}^{-1}\big(\overline{\alpha}_1\, \Phi_{\mathscr{W}}(w_1) \circ T + \overline{\alpha}_2\, \Phi_{\mathscr{W}}(w_2) \circ T\big) \\
&= \alpha_1\, \Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(w_1) \circ T\big) + \alpha_2\, \Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(w_2) \circ T\big) \\
&= \alpha_1\, T^* w_1 + \alpha_2\, T^* w_2.
\end{aligned}
$$

Thus, $T^* \in \mathscr{L}(\mathscr{W}, \mathscr{V})$.

Next we will deduce the most important property of T^*. Recall that from the definition of $\Phi_{\mathscr{V}}^{-1}$ we have that, for a fixed $w \in \mathscr{W}$,

$$
T^* w = \Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(w) \circ T\big)
$$

is equivalent to

$$
\big(\Phi_{\mathscr{W}}(w) \circ T\big)(v) = \langle v, T^* w \rangle_{\mathscr{V}} \quad \text{for all} \quad v \in \mathscr{V},
$$

which, in turn, is equivalent to

$$
\big(\Phi_{\mathscr{W}}(w)\big)(Tu) = \langle v, T^* w \rangle_{\mathscr{V}} \quad \text{for all} \quad v \in \mathscr{V}.
$$

From the definition of $\Phi_{\mathscr{V}}$ the last statement is equivalent to

$$\langle Tv, w\rangle_{\mathscr{W}} = \langle v, T^* w\rangle_{\mathscr{V}} \quad \text{for all} \quad v \in \mathscr{V}.$$

The reasoning above proves the following proposition.

Proposition 4.2. *Let $\bigl(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\bigr)$ and $\bigl(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\bigr)$ be two finite dimensional vector spaces over the same scalar field $\mathbb{F}$ and with positive definite inner products. Let $T \in \mathscr{L}(\mathscr{V},\mathscr{W})$ and $S \in \mathscr{L}(\mathscr{W},\mathscr{V})$. Then $S = T^*$ if and only if*

$$\langle Tv, w\rangle_{\mathscr{W}} = \langle v, Sw\rangle_{\mathscr{V}} \quad \textit{for all} \quad v \in \mathscr{V}, w \in \mathscr{W}. \tag{12}$$

5 Properties of the adjoint operator

Theorem 5.1. *Let $\bigl(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\bigr)$ and $\bigl(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\bigr)$ be two finite dimensional vector spaces over the same scalar field $\mathbb{F}$ and with positive definite inner products. The following statements hold.*

(a) *The adjoint mapping*

$$^* : \mathscr{L}(\mathscr{V},\mathscr{W}) \to \mathscr{L}(\mathscr{W},\mathscr{V})$$

is an anti-linear involution. The inverse of this mapping is the adjoint mapping from $\mathscr{L}(\mathscr{W},\mathscr{V})$ to $\mathscr{L}(\mathscr{V},\mathscr{W})$.

(b) *For $T \in \mathscr{L}(\mathscr{V},\mathscr{W})$ we have*

(i) $\operatorname{nul}(T^*) = (\operatorname{ran} T)^\perp$.
(ii) $\operatorname{ran}(T^*) = (\operatorname{nul} T)^\perp$.
(iii) $\operatorname{nul}(T) = (\operatorname{ran} T^*)^\perp$.
(iv) $\operatorname{ran}(T) = (\operatorname{nul} T^*)^\perp$.

(c) *$T \in \mathscr{L}(\mathscr{V},\mathscr{W})$ is invertible if and only if T^* is invertible; in this case $\bigl(T^{-1}\bigr)^* = \bigl(T^*\bigr)^{-1}$.*

(d) *Let $\mathscr{B}$ and $\mathscr{C}$ be orthonormal bases of $\bigl(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\bigr)$ and $\bigl(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\bigr)$, respectively, and let $T \in \bigl(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\bigr)$. Then $\mathsf{M}_{\mathscr{B}}^{\mathscr{C}}(T^*)$ is the conjugate transpose of the matrix $\mathsf{M}_{\mathscr{C}}^{\mathscr{B}}(T)$.*

Theorem 5.2. *Let $\bigl(\mathscr{U}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{U}}\bigr)$, $\bigl(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\bigr)$ and $\bigl(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\bigr)$ be three finite dimensional vector space over the same scalar field $\mathbb{F}$ and with positive definite inner products. Let $S \in \mathscr{L}(\mathscr{U},\mathscr{V})$ and $T \in \mathscr{L}(\mathscr{V},\mathscr{W})$. Then $(TS)^* = S^*T^*$.*

Lemma 5.3. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$. Let $\mathscr{U}$ be a subspace of $\mathscr{V}$ and let $T \in \mathscr{L}(\mathscr{V})$. The subspace $\mathscr{U}$ is invariant under T if and only if the subspace $\mathscr{U}^{\perp}$ is invariant under T^*.*

Proof. By the definition of adjoint we have

$$\langle Tu, v\rangle = \langle u, T^*v\rangle \tag{13}$$

for all $u, v \in \mathscr{V}$. Assume $T\mathscr{U} \subset \mathscr{U}$. From (13) we get

$$0 = \langle Tu, v\rangle = \langle u, T^*v\rangle \qquad \forall\, u \in \mathscr{U} \quad \text{and} \quad \forall v \in \mathscr{U}^{\perp}.$$

Therefore, $T^*v \in \mathscr{U}^{\perp}$ for all $v \in \mathscr{U}^{\perp}$. This proves "only if" part.

The proof of the "if" part is similar. □

6 Self-adjoint and normal operators

Definition 6.1. Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$. An operator $T \in \mathscr{L}(\mathscr{V})$ is said to be *self-adjoint* if $T = T^*$. An operator $T \in \mathscr{L}(\mathscr{V})$ is said to be *normal* if $TT^* = T^*T$.

Proposition 6.2. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$. All eigenvalues of a self-adjoint $T \in \mathscr{L}(\mathscr{V})$ are real.*

In the rest of this section we will consider only scalar fields $\mathbb{F}$ which contain the imaginary unit i.

Proposition 6.3. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$. Let $T \in \mathscr{L}(\mathscr{V})$. Then $T = 0$ if and only if $\langle Tv, v\rangle = 0$ for all $v \in \mathscr{V}$.*

Proposition 6.4. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$. An operator $T \in \mathscr{L}(\mathscr{V})$ is self-adjoint if and only if $\langle Tv, v\rangle \in \mathbb{R}$ for all $v \in \mathscr{V}$.*

Theorem 6.5. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$. An operator $T \in \mathscr{L}(\mathscr{V})$ is normal if and only if $\|Tv\| = \|T^*v\|$ for all $v \in \mathscr{V}$.*

Corollary 6.6. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$, let $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$ and let $T \in \mathscr{L}(\mathscr{V})$ be normal. Then $\lambda \in \mathbb{C}$ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^* and*

$$\operatorname{nul}\bigl(T^* - \overline{\lambda} I\bigr) = \operatorname{nul}(T - \lambda I).$$

7 The Spectral Theorem

In the rest of the notes we will consider only the scalar field $\mathbb{C}$.

Theorem 7.1 (Theorem 7.9)**.** *Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$ and $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$. Let $T \in \mathscr{L}(\mathscr{V})$. Then $\mathscr{V}$ has an orthonormal basis which consists of eigenvectors of T if and only if T is normal. In other words, T is normal if and only if there exists an orthonormal basis $\mathscr{B}$ of $\mathscr{V}$ such that $\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T)$ is a diagonal matrix.*

Proof. Let $n = \dim(\mathscr{V})$. Assume that T is normal. By Corollary **??** there exists an orthonormal basis $\mathscr{B} = \{u_1, \dots, u_n\}$ of $\mathscr{V}$ such that $\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T)$ is upper-triangular. That is,

$$\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T) = \begin{bmatrix} \langle Tu_1, u_1\rangle & \langle Tu_2, u_1\rangle & \cdots & \langle Tu_n, u_1\rangle \\ 0 & \langle Tu_2, u_2\rangle & \cdots & \langle Tu_n, u_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \langle Tu_n, u_n\rangle \end{bmatrix}, \tag{14}$$

or, equivalently,

$$Tu_k = \sum_{j=1}^{k} \langle Tu_k, u_j\rangle u_j \quad \text{for all} \quad k \in \{1, \dots, n\}. \tag{15}$$

By Theorem 5.1(d) we have

$$\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T^*) = \begin{bmatrix} \overline{\langle Tu_1, u_1\rangle} & 0 & \cdots & 0 \\ \overline{\langle Tu_2, u_1\rangle} & \overline{\langle Tu_2, u_2\rangle} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\langle Tu_n, u_1\rangle} & \overline{\langle Tu_n, u_2\rangle} & \cdots & \overline{\langle Tu_n, u_n\rangle} \end{bmatrix}.$$

Consequently,

$$T^*u_k = \sum_{j=k}^{n} \overline{\langle Tu_j, u_k\rangle} u_j \quad \text{for all} \quad k \in \{1, \dots, n\}. \tag{16}$$

Since T is normal, Theorem 6.5 implies

$$\|Tu_k\|^2 = \|T^*u_k\|^2 \quad \text{for all} \quad k \in \{1, \dots, n\}.$$

Together with (15) and (16) the last identities become

$$\sum_{j=1}^{k} \bigl|\langle Tu_k, u_j\rangle\bigr|^2 = \sum_{j=k}^{n} \bigl|\overline{\langle Tu_j, u_k\rangle}\bigr|^2 \quad \text{for all} \quad k \in \{1,\dots,n\},$$

or, equivalently,

$$\sum_{j=1}^{k} \bigl|\langle Tu_k, u_j\rangle\bigr|^2 = \sum_{j=k}^{n} \bigl|\langle Tu_j, u_k\rangle\bigr|^2 \quad \text{for all} \quad k \in \{1,\dots,n\}. \tag{17}$$

The equality in (17) corresponding to $k = 1$ reads

$$\bigl|\langle Tu_1, u_1\rangle\bigr|^2 = \bigl|\langle Tu_1, u_1\rangle\bigr|^2 + \sum_{j=2}^{n} \bigl|\langle Tu_j, u_1\rangle\bigr|^2,$$

which implies

$$\langle Tu_j, u_1\rangle = 0 \quad \text{for all} \quad j \in \{2,\dots,n\} \tag{18}$$

In other words we have proved that the off-diagonal entries in the first row of the upper triangular matrix $\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T)$ in (14) are all zero.

Substituting the value $\langle Tu_2, u_1\rangle = 0$ (from (18)) in the equality in (17) corresponding to $k = 2$ reads we get

$$\bigl|\langle Tu_2, u_2\rangle\bigr|^2 = \bigl|\langle Tu_2, u_2\rangle\bigr|^2 + \sum_{j=3}^{n} \bigl|\langle Tu_j, u_2\rangle\bigr|^2,$$

which implies

$$\langle Tu_j, u_2\rangle = 0 \quad \text{for all} \quad j \in \{3,\dots,n\} \tag{19}$$

In other words we have proved that the off-diagonal entries in the second row of the upper triangular matrix $\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T)$ in (14) are all zero.

Repeating this reasoning $n-2$ more times would prove that all the off-diagonal entries of the upper triangular matrix $\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T)$ in (14) are zero. That is, $\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T)$ is a diagonal matrix.

To prove the converse, assume that there exists an orthonormal basis $\mathscr{B} = \{u_1,\dots,u_n\}$ of $\mathscr{V}$ which consists of eigenvectors of T. That is, for some $\lambda_j \in \mathbb{C}$,

$$Tu_j = \lambda_j u_j \quad \text{for all} \quad j \in \{1,\dots,n\},$$

Then, for arbitrary $v \in \mathscr{V}$ we have

$$Tv = T\left(\sum_{j=1}^{n} \langle v, u_j\rangle u_j\right) = \sum_{j=1}^{n} \langle v, u_j\rangle Tu_j = \sum_{j=1}^{n} \lambda_j \langle v, u_j\rangle u_j. \tag{20}$$

Therefore, for arbitrary $k \in \{1, \ldots, n\}$ we have

$$\langle Tv, u_k \rangle = \lambda_k \langle v, u_k \rangle. \tag{21}$$

Now we calculate

$$\begin{aligned}
T^*Tv &= \sum_{j=1}^{n} \langle T^*Tv, u_j \rangle u_j \\
&= \sum_{j=1}^{n} \langle Tv, Tu_j \rangle u_j \\
&= \sum_{j=1}^{n} \langle Tv, Tu_j \rangle u_j \\
&= \sum_{j=1}^{n} \overline{\lambda}_j \langle Tv, u_j \rangle u_j \\
&= \sum_{j=1}^{n} \lambda_j \overline{\lambda}_j \langle v, u_j \rangle u_j.
\end{aligned}$$

Similarly,

$$\begin{aligned}
TT^*v &= T\left(\sum_{j=1}^{n} \langle T^*v, u_j \rangle u_j\right) \\
&= \sum_{j=1}^{n} \langle v, Tu_j \rangle Tu_j \\
&= \sum_{j=1}^{n} \langle v, \lambda_j u_j \rangle \lambda_j u_j \\
&= \sum_{j=1}^{n} \lambda_j \overline{\lambda}_j \langle v, u_j \rangle u_j.
\end{aligned}$$

Thus, we proved $T^*Tv = TT^*v$, that is, T is normal. □

A different proof of the "only if" part of the spectral theorem for normal operators follows. In this proof we use δ_{ij} to represent the Kronecker delta function; that is, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

Proof. Set $n = \dim \mathscr{V}$. We first prove "only if" part. Assume that T is normal. Set

$$\mathbb{K} = \left\{ k \in \{1,\ldots,n\} \; : \; \begin{array}{l} \exists\, w_1,\ldots,w_k \in \mathscr{V} \quad \text{and} \quad \exists\, \lambda_1,\ldots,\lambda_k \in \mathbb{C} \\ \text{such that } \langle w_i, w_j \rangle = \delta_{ij} \text{ and } Tw_j = \lambda_j w_j \\ \text{for all} \quad i,j \in \{1,\ldots,k\} \end{array} \right\}$$

Clearly $1 \in \mathbb{K}$. Since $\mathbb{K}$ is finite, $m = \max \mathbb{K}$ exists. Clearly, $m \leq n$.

Next we will prove that $k \in \mathbb{K}$ and $k < n$ implies that $k+1 \in \mathbb{K}$. Assume $k \in \mathbb{K}$ and $k < n$. Let $w_1,\ldots,w_k \in \mathscr{V}$ and $\lambda_1,\ldots,\lambda_k \in \mathbb{C}$ be such that $\langle w_i, w_j \rangle = \delta_{ij}$ and $Tw_j = \lambda_j w_j$ for all $i,j \in \{1,\ldots,k\}$. Set

$$\mathscr{W} = \operatorname{span}\{w_1,\ldots,w_k\}.$$

Since $w_1,\ldots,w_k$ are eigenvectors of T we have $T\mathscr{W} \subseteq \mathscr{W}$. By Lemma 5.3, $T^*\big(\mathscr{W}^\perp\big) \subseteq \mathscr{W}^\perp$. Thus, $T^*|_{\mathscr{W}^\perp} \in \mathscr{L}\big(\mathscr{W}^\perp\big)$. Since $\dim \mathscr{W} = k < n$ we have $\dim\big(\mathscr{W}^\perp\big) = n - k \geq 1$. Since $\mathscr{W}^\perp$ is a complex vector space the operator $T^*|_{\mathscr{W}^\perp}$ has an eigenvalue μ with the corresponding unit eigenvector u. Clearly, $u \in \mathscr{W}^\perp$ and $T^*u = \mu u$. Since T^* is normal, Corollary 6.6 yields that $Tu = \overline{\mu}u$. Since $u \in \mathscr{W}^\perp$ and $Tu = \overline{\mu}u$, setting $w_{k+1} = u$ and $\lambda_{k+1} = \overline{\mu}$ we have

$$\langle w_i, w_j \rangle = \delta_{ij} \quad \text{and} \quad Tw_j = \lambda_j w_j \quad \text{for all} \quad i,j \in \{1,\ldots,k,k+1\}.$$

Thus $k+1 \in \mathbb{K}$. Consequently, $k < m$. Thus, for $k \in \mathbb{K}$, we have proved the implication

$$k < n \qquad \Rightarrow \qquad k < m.$$

The contrapositive of this implication is: For $k \in \mathbb{K}$, we have

$$k \geq m \qquad \Rightarrow \qquad k \geq n.$$

In particular, for $m \in \mathbb{K}$ we have $m = m$ implies $m \geq n$. Since $m \leq n$ is also true, this proves that $m = n$. That is, $n \in \mathbb{K}$. This implies that there exist $u_1,\ldots,u_n \in \mathscr{V}$ and $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$ such that $\langle u_i, u_j \rangle = \delta_{ij}$ and $Tu_j = \lambda_j u_j$ for all $i,j \in \{1,\ldots,n\}$.

Since $u_1,\ldots,u_n$ are orthonormal, they are linearly independent. Since $n = \dim \mathscr{V}$, it turns out that $u_1,\ldots,u_n$ form a basis of $\mathscr{V}$. This completes the proof. $\square$

8 Invariance under a normal operator

Theorem 8.1 (Theorem 7.18)**.** *Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$. Let $\langle\cdot,\cdot\rangle$ be a positive definite inner product on $\mathscr{V}$. Let $T \in \mathscr{L}(\mathscr{V})$ be normal and let $\mathscr{U}$ be a subspace of $\mathscr{V}$. Then*

$$T\mathscr{U} \subseteq \mathscr{U} \quad \Leftrightarrow \quad T\mathscr{U}^{\perp} \subseteq \mathscr{U}^{\perp}$$

(Recall that we have previously proved that for any $T \in \mathscr{L}(\mathscr{V})$, $T\mathscr{U} \subseteq \mathscr{U} \Leftrightarrow T^*\mathscr{U}^{\perp} \subseteq \mathscr{U}^{\perp}$. Hence if T is normal, showing that any one of $\mathscr{U}$ or $\mathscr{U}^{\perp}$ is invariant under either T or T^* implies that the rest are, also.)

Proof. Assume $T\mathscr{U} \subseteq \mathscr{U}$. We know $\mathscr{V} = \mathscr{U} \oplus \mathscr{U}^{\perp}$. Let $u_1, \ldots, u_m$ be an orthonormal basis of $\mathscr{U}$ and $u_{m+1}, \ldots, u_n$ be an orthonormal basis of $\mathscr{U}^{\perp}$. Then $u_1, \ldots, u_n$ is an orthonormal basis of $\mathscr{V}$. If $j \in \{1, \ldots, m\}$ then $u_j \in \mathscr{U}$, so $Tu_j \in \mathscr{U}$. Hence

$$Tu_j = \sum_{k=1}^{m} \langle Tu_j, u_k\rangle u_k.$$

Also, clearly,

$$T^*u_j = \sum_{k=1}^{n} \langle T^*u_j, u_k\rangle u_k.$$

By normality of T we have $\|Tu_j\|^2 = \|T^*u_j\|^2$ for all $j \in \{1, \ldots, m\}$. Starting with this, we calculate

$$\begin{aligned}
\sum_{j=1}^{m} \|Tu_j\|^2 &= \sum_{j=1}^{m} \|T^*u_j\|^2 \\
\boxed{\text{Pythag. thm.}} &= \sum_{j=1}^{m}\sum_{k=1}^{n} |\langle T^*u_j, u_k\rangle|^2 \\
\boxed{\text{group terms}} &= \sum_{j=1}^{m}\sum_{k=1}^{m} |\langle T^*u_j, u_k\rangle|^2 + \sum_{j=1}^{m}\sum_{k=m+1}^{n} |\langle T^*u_j, u_k\rangle|^2 \\
\boxed{\text{def. of } T^*} &= \sum_{j=1}^{m}\sum_{k=1}^{m} |\langle u_j, Tu_k\rangle|^2 + \sum_{j=1}^{m}\sum_{k=m+1}^{n} |\langle T^*u_j, u_k\rangle|^2 \\
\boxed{|\alpha| = |\overline{\alpha}|} &= \sum_{j=1}^{m}\sum_{k=1}^{m} |\langle Tu_k, u_j\rangle|^2 + \sum_{j=1}^{m}\sum_{k=m+1}^{n} |\langle T^*u_j, u_k\rangle|^2
\end{aligned}$$

$$\boxed{\text{order of sum.}} = \sum_{k=1}^{m}\sum_{j=1}^{m} |\langle Tu_k, u_j\rangle|^2 + \sum_{j=1}^{m}\sum_{k=m+1}^{n} |\langle T^*u_j, u_k\rangle|^2$$

$$\boxed{\text{Pythag. thm.}} = \sum_{k=1}^{m} \|Tu_k\|^2 + \sum_{j=1}^{m}\sum_{k=m+1}^{n} |\langle T^*u_j, u_k\rangle|^2.$$

From the above equality we deduce that $\sum_{j=1}^{m}\sum_{k=m+1}^{n} |\langle T^*u_j, u_k\rangle|^2 = 0$. As each term is nonnegative, we conclude that $|\langle T^*u_j, u_k\rangle|^2 = |\langle u_j, Tu_k\rangle|^2 = 0$, that is,

$$\langle u_j, Tu_k\rangle = 0 \quad \text{for all} \quad j \in \{1, \ldots, m\},\ k \in \{m+1, \ldots, n\}. \tag{22}$$

Let now $w \in \mathscr{U}^\perp$ be arbitrary. Then

$$\begin{aligned} Tw &= \sum_{j=1}^{n} \langle Tw, u_j\rangle u_j \\ &= \sum_{j=1}^{n} \left\langle \sum_{k=m+1}^{n} \langle w, u_k\rangle Tu_k, u_j \right\rangle u_j \\ &= \sum_{j=1}^{n}\sum_{k=m+1}^{n} \langle w, u_k\rangle \langle Tu_k, u_j\rangle u_j \\ \boxed{\text{by (22)}} &= \sum_{j=m+1}^{n}\sum_{k=m+1}^{n} \langle w, u_k\rangle \langle Tu_k, u_j\rangle u_j \end{aligned}$$

Hence $Tw \in \mathscr{U}^\perp$, that is $T\mathscr{U}^\perp \subseteq \mathscr{U}^\perp$. □

A different proof follows. The proof below uses the property of polynomials that for arbitrary distinct $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ and arbitrary $\beta_1, \ldots, \beta_m \in \mathbb{C}$ there exists a polynomial $p(z) \in \mathbb{C}[z]_{<m}$ such that $p(\alpha_j) = \beta_j$, $j \in \{1, \ldots, m\}$.

Proof. Assume T is normal. Then there exists an orthonormal basis $\{u_1, \ldots, u_n\}$ and $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{C}$ such that

$$Tu_j = \lambda_j u_j \quad \text{for all} \quad j \in \{1, \ldots, n\}.$$

Consequently,

$$T^*u_j = \overline{\lambda}_j u_j \quad \text{for all} \quad j \in \{1, \ldots, n\}.$$

Let v be arbitrary in $\mathscr{V}$. Applying T and T^* to the expansion of v in the basis vectors $\{u_1, \dots, u_n\}$ we obtain

$$Tv = \sum_{j=1}^{n} \lambda_j \langle v, u_j \rangle u_j$$

and

$$T^* v = \sum_{j=1}^{n} \overline{\lambda_j} \langle v, u_j \rangle u_j.$$

Let $p(z) = a_0 + a_1 z + \cdots + a_m z^m \in \mathbb{C}[z]$ be such that

$$p(\lambda_j) = \overline{\lambda}_j, \quad \text{for all} \quad j \in \{1, \dots, n\}.$$

Clearly, for all $j \in \{1, \dots, n\}$ we have

$$p(T) u_j = p(\lambda_j) u_j = \overline{\lambda}_j u_j = T^* u_j.$$

Therefore $p(T) = T^*$.

Now assume $T\mathscr{U} \subseteq \mathscr{U}$. Then $T^k \mathscr{U} \subseteq \mathscr{U}$ for all $k \in \mathbb{N}$ and also $\alpha T \mathscr{U} \subseteq \mathscr{U}$ for all $\alpha \in \mathbb{C}$. Hence $p(T)\mathscr{U} = T^* \mathscr{U} \subseteq \mathscr{U}$. The theorem follows from Lemma 5.3. $\square$

Lastly we review the proof in the book. This proof is in essence very similar to the first proof. It brings up a matrix representation of T for easier visualization of what we are doing.

Proof. Assume $T\mathscr{U} \subseteq \mathscr{U}$. By Lemma 5.3 $T^*(\mathscr{U}^\perp) \subseteq \mathscr{U}^\perp$.

Now $\mathscr{V} = \mathscr{U} \oplus \mathscr{U}^\perp$. Let $n = \dim(\mathscr{V})$. Let $\{u_1, \dots, u_m\}$ be an orthonormal basis of $\mathscr{U}$ and $\{u_{m+1}, \dots, u_n\}$ be an orthonormal basis of $\mathscr{U}^\perp$. Then $\mathscr{B} = \{u_1, \dots, u_n\}$ is an orthonormal basis of $\mathscr{V}$. Since $Tu_j \in \mathscr{U}$ for all $j \in \{1, \dots, m\}$ we have

$$\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T) = \begin{array}{c} \\ u_1 \\ \vdots \\ u_m \\ u_{m+1} \\ \vdots \\ u_n \end{array} \begin{array}{c} \begin{array}{ccc|ccc} Tu_1 & \cdots & Tu_m & Tu_{m+1} & \cdots & Tu_n \end{array} \\ \left[\begin{array}{ccc|c} \langle Tu_1, u_m \rangle & \cdots & \langle Tu_m, u_1 \rangle & \\ \vdots & \ddots & \vdots & B \\ \langle Tu_1, u_1 \rangle & \cdots & \langle Tu_m, u_m \rangle & \\ \hline & & & \\ & 0 & & C \\ & & & \end{array} \right] \end{array}$$

Here we added the basis vectors and their images around the matrix to emphasize that a vector Tu_k in the zeroth row is expended as a linear combination of the vectors in the zeroth column with the coefficients given in the k-th column of the matrix.

For $j \in \{1, \dots, m\}$ we have $Tu_j = \sum_{k=1}^{m} \langle Tu_j, u_k \rangle u_k$. By Pythagorean Theorem $\|Tu_j\|^2 = \sum_{k=1}^{m} |\langle Tu_j, u_k \rangle|^2$ and $\|T^*u_j\|^2 = \sum_{k=1}^{n} |\langle T^*u_j, u_k \rangle|^2$. Since T is normal, $\sum_{j=1}^{m} \|Tu_j\|^2 = \sum_{j=1}^{m} \|T^*u_j\|^2$. Now we have

$$\begin{aligned}\sum_{j=1}^{m}\sum_{k=1}^{m} |\langle Tu_j, u_k \rangle|^2 &= \sum_{j=1}^{m}\sum_{k=1}^{m} |\langle T^*u_j, u_k \rangle|^2 + \sum_{j=1}^{m}\sum_{k=m+1}^{n} |\langle T^*u_j, u_k \rangle|^2 \\ &= \sum_{j=1}^{m}\sum_{k=1}^{m} |\langle Tu_k, u_j \rangle|^2 + \sum_{j=1}^{m}\sum_{k=m+1}^{n} |\langle T^*u_j, u_k \rangle|^2.\end{aligned}$$

Canceling the identical terms we get that the last double sum which consists of the nonnegative terms is equal to 0. Hence $|\langle T^*u_j, u_k \rangle|^2 = |\langle u_j, Tu_k \rangle|^2 = |\langle Tu_k, u_j \rangle|^2$, and thus, $\langle Tu_k, u_j \rangle = 0$ for all $j \in \{1, \dots, m\}$ and for all $k \in \{m+1, \dots, n\}$. This proves that $B = 0$ in the above matrix representation. Therefore, Tu_k is orthogonal to $\mathscr{U}$ for all $k \in \{m+1, \dots, n\}$, which implies $T(\mathscr{U}^\perp) \subseteq \mathscr{U}^\perp$. $\square$

Theorem 8.1 and Lemma 5.3 yield the following corollary.

Corollary 8.2. *Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$. Let $\langle \cdot, \cdot \rangle$ be a positive definite inner product on $\mathscr{V}$. Let $T \in \mathscr{L}(\mathscr{V})$ be normal and let $\mathscr{U}$ be a subspace of $\mathscr{V}$. The following statements are equivalent:*

(a) $T\mathscr{U} \subseteq \mathscr{U}$.
(b) $T(\mathscr{U}^\perp) \subseteq \mathscr{U}^\perp$.
(c) $T^*\mathscr{U} \subseteq \mathscr{U}$.
(d) $T^*(\mathscr{U}^\perp) \subseteq \mathscr{U}^\perp$.

If any of the for above statements are true, then the following statements are true

(e) $\left(T\big|_{\mathscr{U}}\right)^* = T^*\big|_{\mathscr{U}}$.
(f) $\left(T\big|_{\mathscr{U}^\perp}\right)^* = T^*\big|_{\mathscr{U}^\perp}$.
(g) $T\big|_{\mathscr{U}}$ *is a normal operator on* $\mathscr{U}$.
(h) $T\big|_{\mathscr{U}^\perp}$ *is a normal operator on* $\mathscr{U}^\perp$.

9 Polar Decomposition

There are two distinct subsets of $\mathbb{C}$. Those are the set of nonnegative real numbers, denoted by $\mathbb{R}_{\geq 0}$, and the set of complex numbers of modulus 1, denoted by $\mathbb{T}$. An important tool in complex analysis is the polar representation of a complex number: for every $\alpha \in \mathbb{C}$ there exists $r \in \mathbb{R}_{\geq 0}$ and $u \in \mathbb{T}$ such that $\alpha - r\,u$.

In this section we will prove that an analogous statement holds for operators in $\mathscr{L}(\mathscr{V})$, where $\mathscr{V}$ is a finite dimensional vector space over $\mathbb{C}$ with a positive definite inner product. The first step towards proving this analogous result is identifying operators in $\mathscr{L}(\mathscr{V})$ which will play the role of nonnegative real numbers and the role of complex numbers with modulus 1. That is done in the following two definitions.

Definition 9.1. Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$ with a positive definite inner product $\langle\,\cdot\,,\cdot\,\rangle$. An operator $Q \in \mathscr{L}(\mathscr{V})$ is said to be *nonnegative* if $\langle Qv, v\rangle \geq 0$ for all $v \in \mathscr{V}$.

Note that Axler uses the term “positive” instead of nonnegative. We think that nonnegative is more appropriate, since $0_{\mathscr{L}(\mathscr{V})}$ is a nonnegative operator. There is nothing positive about any zero, we think.

Proposition 9.2. *Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$ with a positive definite inner product $\langle\,\cdot\,,\cdot\,\rangle$ and let $T \in \mathscr{L}(\mathscr{V})$. Then T is nonnegative if and only if T is normal and all its eigenvalues are nonnegative.*

Theorem 9.3. *Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$ with a positive definite inner product $\langle\,\cdot\,,\cdot\,\rangle$. Let $Q \in \mathscr{L}(\mathscr{V})$ be a nonnegative operator and let $u_1, \ldots, u_n$ be an orthonormal basis of $\mathscr{V}$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_{\geq 0}$ be such that*

$$Qu_j = \lambda_j u_j \quad \textit{for all} \quad j \in \{1, \ldots, n\}. \tag{23}$$

The following statements are equivalent.

(a) *$S \in \mathscr{L}(\mathscr{V})$ be a nonnegative operator and $S^2 = Q$.*

(b) *For every $\lambda \in \mathbb{R}_{\geq 0}$ we have*

$$\operatorname{nul}(Q - \lambda I) = \operatorname{nul}(S - \sqrt{\lambda} I).$$

(c) *For every $v \in \mathscr{V}$ we have*

$$Sv = \sum_{j=1}^{n} \sqrt{\lambda_j}\langle v, u_j\rangle u_j.$$

Proof. (a) ⇒ (b). We first prove that $\operatorname{nul} Q = \operatorname{nul} S$. Since $Q = S^2$ we have $\operatorname{nul} S \subseteq \operatorname{nul} Q$. Let $v \in \operatorname{nul} Q$, that is, let $Qv = S^2 v = 0$. Then $\langle S^2 v, v\rangle = 0$. Since S is nonnegative it is self-adjoint. Therefore, $\langle S^2 v, v\rangle = \langle Sv, Sv\rangle = \|Sv\|^2$. Hence, $\|Sv\| = 0$, and thus $Sv = 0$. This proves that $\operatorname{nul} Q \subseteq \operatorname{nul} S$ and (b) is proved for $\lambda = 0$.

Let $\lambda > 0$. Then the operator $S + \sqrt{\lambda} I$ is invertible. To prove this, let $v \in \mathscr{V} \setminus \{0_{\mathscr{V}}\}$ be arbitrary. Then $\|v\| > 0$ and therefore

$$\big\langle (S + \sqrt{\lambda} I)v, v\big\rangle = \langle Sv, v\rangle + \sqrt{\lambda}\langle v, v\rangle \geq \sqrt{\lambda}\|v\|^2 > 0.$$

Thus, $v \neq 0$ implies $(S + \sqrt{\lambda} I)v \neq 0$. This proves the injectivity of $S + \sqrt{\lambda} I$.

To prove $\operatorname{nul}(Q - \lambda I) = \operatorname{nul}(S - \sqrt{\lambda} I)$, let $v \in \mathscr{V}$ be arbitrary and notice that $(Q - \lambda I)v = 0$ if and only if $\big(S^2 - \sqrt{\lambda}^2 I\big)v = 0$, which, in turn, is equivalent to

$$\big(S + \sqrt{\lambda} I\big)\big(S - \sqrt{\lambda} I\big)v = 0.$$

Since $S + \sqrt{\lambda} I$ is injective, the last equality is equivalent to $\big(S - \sqrt{\lambda} I\big)v = 0$. This completes the proof of (b).

(b) ⇒ (c). Let $u_1, \ldots, u_n$ be an orthonormal basis of $\mathscr{V}$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_{\geq 0}$ be such that (23) holds. For arbitrary $j \in \{1, \ldots, n\}$ (23) yields $u_j \in \operatorname{nul}(Q - \lambda_j I)$. By (b), $u_j \in \operatorname{nul}(S - \sqrt{\lambda_j} I)$. Thus

$$S u_j = \sqrt{\lambda_j}\, u_j \quad \text{for all} \quad j \in \{1, \ldots, n\}. \tag{24}$$

Let $v = \sum_{j=1}^{n} \langle v, u_j\rangle u_j$ be arbitrary vector in $\mathscr{V}$. Then, the linearity of S and (24) imply the claim in (c).

The implication (c) ⇒ (a) is straightforward. □

The implication (a) ⇒ (c) of Theorem 9.3 yields that for a given nonnegative Q a nonnegative S such that $Q = S^2$ is uniquely determined. The common notation for this unique S is $\sqrt{Q}$.

Definition 9.4. Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$ with a positive definite inner product $\langle\,\cdot\,,\cdot\,\rangle$. An operator $U \in \mathscr{L}(\mathscr{V})$ is said to be *unitary* if $U^* U = I$.

Proposition 9.5. *Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$ with a positive definite inner product $\langle\,\cdot\,,\cdot\,\rangle$ and let $T \in \mathscr{L}(\mathscr{V})$. The following statements are equivalent.*

(a) *T is unitary.*

(b) *For all $u, v \in \mathscr{V}$ we have $\langle Tu, Tv\rangle = \langle u, v\rangle$.*

(c) *For all* $v \in \mathscr{V}$ *we have* $\|Tv\| = \|v\|$.

(d) T *is normal and all its eigenvalues have modulus* 1.

Theorem 9.6 (Polar Decomposition in $\mathscr{L}(\mathscr{V})$, Theorem 7.41)**.** *Let* $\mathscr{V}$ *be a finite dimensional vector space over* $\mathbb{C}$ *with a positive definite inner product* $\langle \cdot, \cdot \rangle$. *For every* $T \in \mathscr{L}(\mathscr{V})$ *there exist a unitary operator* U *in* $\mathscr{L}(\mathscr{V})$ *and a unique nonnegative* $Q \in \mathscr{L}(\mathscr{V})$ *such that* $T = UQ$; U *is unique if and only if* T *is invertible.*

Proof. First, notice that the operator T^*T is nonnegative: for every $v \in \mathscr{V}$ we have

$$\langle T^*Tv, v\rangle = \langle Tv, Tv\rangle = ||Tv||^2 \geq 0.$$

To prove the uniqueness of Q assume that $T = UQ$ with U unitary and Q nonnegative. Then $Q^* = Q$, $U^* = U^{-1}$ and therefore, $T^*T = Q^*U^*UQ = QU^{-1}UQ = Q^2$. Since Q is nonnegative we have $Q = \sqrt{T^*T}$.

Set $Q = \sqrt{T^*T}$. By Theorem 9.3(b) we have $\operatorname{nul} Q = \operatorname{nul}(T^*T)$. Moreover, we have $\operatorname{nul}(T^*T) = \operatorname{nul} T$. The inclusion $\operatorname{nul} T \subseteq \operatorname{nul}(T^*T)$ is trivial. For the converse inclusion notice that $v \in \operatorname{nul}(T^*T)$ implies $T^*Tv = 0$, which yields $\langle T^*Tv, v\rangle = 0$ and thus $\langle Tv, Tv\rangle = 0$. Consequently, $\|Tv\| = 0$, that is $Tv = 0$, yielding $v \in \operatorname{nul} T$. So,

$$\operatorname{nul} Q = \operatorname{nul}(T^*T) = \operatorname{nul} T \tag{25}$$

is proved.

First assume that T is invertible. By (25) and ??, Q is invertible as well. Therefore $T = UQ$ is equivalent to $U = TQ^{-1}$ in this case. Since Q is unique, this proves the uniqueness of U. Set $U = TQ^{-1}$. Since Q is self-adjoint, Q^{-1} is also self-adjoint. Therefore $U^* = Q^{-1}T^*$, yielding $U^*U = Q^{-1}T^*TQ^{-1} = Q^{-1}Q^2Q^{-1} = I$. That is, U is unitary.

Now assume that T is not invertible. By the Nullity-Rank Theorem, $\dim(\operatorname{ran} Q) = \dim(\operatorname{ran} T)$. Since T is not invertible, $\dim(\operatorname{ran} Q) = \dim(\operatorname{ran} T) < \dim \mathscr{V}$, implying that

$$\dim\bigl((\operatorname{ran} Q)^\perp\bigr) = \dim\bigl((\operatorname{ran} T)^\perp\bigr) > 0. \tag{26}$$

We will define $U : \mathscr{V} \to \mathscr{V}$ in two steps. First we define the action of U on $\operatorname{ran} Q$, that is we define the operator $U_r : \operatorname{ran} Q \to \operatorname{ran} T$, then we define an operator $U_p : (\operatorname{ran} Q)^\perp \to (\operatorname{ran} T)^\perp$.

We define $U_r : \operatorname{ran} Q \to \operatorname{ran} T$ in the following way: Let $u \in \operatorname{ran} Q$ be arbitrary and let $x \in \mathscr{V}$ be such that $u = Qx$. Then we set

$$U_r u = Tx.$$

First we need to show that U_r is well defined. Let $x_1, x_2 \in \mathscr{V}$ be such that $u = Qx_1 = Qx_2$. Then, $x_1 - x_2 \in \operatorname{nul} Q$. Since $\operatorname{nul} Q = \operatorname{nul} T$, we thus have $x_1 - x_2 \in \operatorname{nul} T$. Consequently, $Tx_1 = Tx_2$.

To prove that U_r is angle-preserving, let $u_1, u_2 \in \operatorname{ran} Q$ be arbitrary and let $x_1, x_1 \in \mathscr{V}$ be such that $u_1 = Qx_1$ and $u_2 = Qx_2$ and calculate

$$\begin{aligned}
\langle U_r u_1, U_r u_2 \rangle &= \big\langle U_r(Qx_1), U_r(Qx_2) \big\rangle \\
\boxed{\text{by definition of } U_r} &= \langle Tx_1, Tx_2 \rangle \\
\boxed{\text{by definition of adjoint}} &= \langle T^*Tx_1, x_2 \rangle \\
\boxed{\text{by definition of } Q} &= \langle Q^2 x_1, x_2 \rangle \\
\boxed{\text{since } Q \text{ is self-adjoint}} &= \langle Qx_1, Qx_2 \rangle \\
\boxed{\text{by definition of } x_1, x_2} &= \langle u_1, u_2 \rangle
\end{aligned}$$

Thus $U_r : \operatorname{ran}(Q) \to \operatorname{ran}(T)$ is angle-preserving.

Next we define an angle-preserving operator

$$U_p : (\operatorname{ran} Q)^{\perp} \to (\operatorname{ran} T)^{\perp}.$$

By (26), we can set

$$m = \dim\big((\operatorname{ran} Q)^{\perp}\big) = \dim\big((\operatorname{ran} T)^{\perp}\big) > 0.$$

Let $e_1, \dots, e_m$ be an orthonormal basis on $(\operatorname{ran} Q)^{\perp}$ and let $f_1, \dots, f_m$ be an orthonormal basis on $(\operatorname{ran} T)^{\perp}$. For arbitrary $w \in (\operatorname{ran} P)^{\perp}$ define

$$U_p w = U_p\left(\sum_{j=1}^{m} \langle w, e_j \rangle e_j\right) = \sum_{j=1}^{m} \langle w, e_j \rangle f_j.$$

Then, for $w_1, w_2 \in (\operatorname{ran} Q)^{\perp}$ we have

$$\langle U_p w_1, U_p w_2 \rangle = \left\langle \sum_{i=1}^{m} \langle w_1, e_i \rangle f_i, \sum_{j=1}^{m} \langle w_2, e_j \rangle f_j \right\rangle = \sum_{j=1}^{m} \langle w_1, e_j \rangle \overline{\langle w_2, e_j \rangle} = \langle w_1, w_2 \rangle.$$

Hence U_p is angle-preserving on $(\operatorname{ran} Q)^{\perp}$.

Since the orthomormal bases in the definition of U_p were arbitrary and since $m > 0$, the operator U_p is not unique.

Finally we define $U : \mathscr{V} \to \mathscr{V}$ as a direct sum of U_ra dna U_p. Recall that

$$\mathscr{V} = (\operatorname{ran} Q) \oplus (\operatorname{ran} Q)^{\perp}.$$

Let $v \in \mathscr{V}$ be arbitrary. Then there exist unique $u \in (\operatorname{ran} Q)$ and $w \in (\operatorname{ran} Q)^{\perp}$ such that $v = u + w$. Set

$$Uv = U_r u + U_p w.$$

We claim that U is angle-preserving. Let $v_1, v_2 \in \mathscr{V}$ be arbitrary and let $v_i = u_i + w_i$ with $u_i \in (\operatorname{ran} Q)$ and $w_i \in (\operatorname{ran} Q)^{\perp}$, $i \in \{1, 2\}$. Notice that

$$\langle v_1, v_2 \rangle = \langle u_1 + w_1, u_2 + w_2 \rangle = \langle u_1, u_2 \rangle + \langle w_1, w_2 \rangle, \tag{27}$$

since u_1, u_2 are orthogonal to w_1, w_2. Similarly

$$\langle U_r u_1 + U_p w_1, U_r u_2 + U_p w_2 \rangle = \langle U_r u_1, U_r u_2 \rangle + \langle U_p w_1, U_p w_2 \rangle, \tag{28}$$

since $U_r u_1, U_r u_2 \in (\operatorname{ran} T)$ and $U_p w_1, U_p w_2 \in (\operatorname{ran} T)^{\perp}$. Now we calculate, starting with the definition of U,

$$\begin{aligned}
\langle Uv_1, Uv_2 \rangle &= \langle U_r u_1 + U_p w_1, U_r u_2 + U_p w_2 \rangle \\
\boxed{\text{by (28)}} &= \langle U_r u_1, U_r u_2 \rangle + \langle U_p w_1, U_p w_2 \rangle \\
\boxed{U_r \text{ and } U_p \text{ are angle-preserving}} &= \langle u_1, u_2 \rangle + \langle w_1, w_2 \rangle \\
\boxed{\text{by (27)}} &= \langle v_1, v_2 \rangle.
\end{aligned}$$

Hence U is angle-preserving and by Proposition 9.5 we have that U is unitary.

Finally we show that $T = UQ$. Let $v \in \mathscr{V}$ be arbitrary. Then $Qv \in \operatorname{ran} Q$. By definitions of U and U_r we have

$$UQv = U_r Qv = Tv.$$

Thus $T = UQ$, where U is unitary and Q is nonnegative. □

Theorem 9.7 (Singular-Value Decomposition, Theorem 7.46)**.** *Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$ with a positive definite inner product $\langle \cdot, \cdot \rangle$ and let $T \in \mathscr{L}(\mathscr{V})$. Then there exist orthonormal bases $\mathscr{B} = \{u_1, \ldots, u_n\}$ and $\mathscr{C} = \{w_1, \ldots, w_n\}$ and nonnegative scalars $\sigma_1, \ldots, \sigma_n$ such that for every $v \in \mathscr{V}$ we have*

$$Tv = \sum_{j=1}^{n} \sigma_j \langle v, u_j \rangle w_j. \tag{29}$$

In other words, there exist orthonormal bases $\mathscr{B}$ and $\mathscr{C}$ such that the matrix $\mathsf{M}_{\mathscr{C}}^{\mathscr{B}}(T)$ is diagonal with nonnegative entries on the diagonal.

Proof. Let $T = UQ$ be a polar decomposition of T, that is let U be unitary and $Q = \sqrt{T^*T}$. Since Q is nonnegative, it is normal with nonnegative eigenvalues. By the spectral theorem, there exists an orthonormal basis $\{u_1, \dots, u_n\}$ of $\mathscr{V}$ and nonnegative scalars $\sigma_1, \dots, \sigma_n$ such that

$$Qu_j = \sigma_j u_j \quad \text{for all} \quad j \in \{1, \dots, n\}. \tag{30}$$

Since $\{u_1, \dots, u_n\}$ is an orthonormal basis, for arbitrary $v \in \mathscr{V}$ we have

$$v = \sum_{j=1}^{n} \langle v, u_j \rangle u_j. \tag{31}$$

Applying Q to (31), using its linearity and (30) we get

$$Qv = \sum_{j=1}^{n} \sigma_j \langle v, u_j \rangle u_j. \tag{32}$$

Applying U to (32) and using its linearity we get

$$UQv = \sum_{j=1}^{n} \sigma_j \langle v, u_j \rangle U u_j. \tag{33}$$

Set $w_j = Uu_j$, $j \in \{1, \dots, n\}$. This definition and the fact that U is angle-preserving yield

$$\langle w_i, w_j \rangle = \langle Uu_i, Uu_j \rangle = \langle u_i, u_j \rangle = \delta_{ij}.$$

Thus $\{w_1, \dots, w_n\}$ is an orthonormal basis. Substituting $w_j = Uu_j$ in (33) and using $T = UQ$ we get (29). ☐

The values $\sigma_1, \dots, \sigma_n$ from Theorem 9.7, which are in fact the eigenvalues of $\sqrt{T^*T}$, are called *singular values* of T.