MATH 504

Assignment 1
January 17, 2020

Name ____________________

Problem 1. Let $\mathcal{V} = (-1,1)$ and let $\mathbb{F} = \mathbb{R}$. Define the addition and the scalar multiplication in $\mathcal{V}$ by: For all $u, v \in \mathcal{V}$ and all $\alpha \in \mathbb{R}$ set

$$u \oplus v = \frac{u+v}{1+uv}, \qquad \alpha \odot v = \frac{(1+v)^\alpha - (1-v)^\alpha}{(1+v)^\alpha + (1-v)^\alpha}.$$

Prove that $\mathcal{V}$ with the vector addition $\oplus$ and the scaling $\odot$ is a vector space over $\mathbb{R}$.

Problem 2. Consider the vector space $\mathbb{R}^\mathbb{R}$ of all real valued functions defined on $\mathbb{R}$. This vector space is considered over the field $\mathbb{R}$. The purpose of this exercise is to study some special subspaces of the vector space $\mathbb{R}^\mathbb{R}$. Let γ be an arbitrary (fixed) real number. Consider the set

$$\mathcal{S}_\gamma := \Bigl\{ f \in \mathbb{R}^\mathbb{R} : \exists\, a, b \in \mathbb{R} \ \text{ such that } \ f(t) = a \sin(\gamma t + b) \ \ \forall\, t \in \mathbb{R} \Bigr\}.$$

(a) Do you see exceptional values for γ for which the set $\mathcal{S}_\gamma$ is particularly simple?

(b) Prove that $\mathcal{S}_\gamma$ is a subspace of $\mathbb{R}^\mathbb{R}$.

(c) For each $\gamma \in \mathbb{R}$ find a basis for $\mathcal{S}_\gamma$. Plot the function $\gamma \mapsto \dim \mathcal{S}_\gamma$.

Problem 3. Let D be a nonempty set and $\mathbb{F}$ a scalar field. Let $\mathbb{F}^D$ be a vector space of all functions defined on D with values in $\mathbb{F}$. Let $\varphi : D \to D$ be a bijection. Set

$$\mathcal{O} = \bigl\{ f \in \mathbb{F}^D : f\bigl(\varphi(t)\bigr) = -f(t) \ \forall t \in D \bigr\},$$
$$\mathcal{E} = \bigl\{ f \in \mathbb{F}^D : f\bigl(\varphi(t)\bigr) = f(t) \ \forall t \in D \bigr\}.$$

(a) Prove that $\mathcal{O}$ and $\mathcal{E}$ are subspaces of $\mathbb{F}^D$.

(b) Prove $\mathcal{O} \cap \mathcal{E} = \bigl\{ 0_{\mathbb{F}^D} \bigr\}$.

(c) Characterize the functions in the set $\mathcal{O} + \mathcal{E}$.

(d) Find a necessary and sufficient condition on $\varphi : D \to D$ for the equality $\mathbb{F}^D = \mathcal{O} + \mathcal{E}$ to hold.

Note: This problem is inspired by the concepts of odd and even functions encountered in a precalculus class. In this precalculus setting $D = \mathbb{R}$, $\mathbb{F} = \mathbb{R}$ and $\varphi(t) = -t, t \in \mathbb{R}$. It would be helpful to work out this problem for this particular case first.

Problem 4. Let $\mathcal{V}$ be a vector space over $\mathbb{F}$. Let $\mathcal{A}$ be a linearly independent subset of $\mathcal{V}$. Let $u \in \mathcal{V}$ be arbitrary. By $u + \mathcal{A}$ we denote the set of vectors $\{u + v : v \in \mathcal{A}\}$.

(a) Prove the following implication. If $w \notin \operatorname{span} \mathcal{A}$, then $w + \mathcal{A}$ is a linearly independent set.

(b) Is the converse of the implication in (a) true? Justify your claim.

(c) Let $\alpha_1, \cdots, \alpha_n \in \mathbb{F}$, let $v_1, \ldots, v_n$ be distinct vectors in $\mathcal{A}$ and let $w = \alpha_1 v_1 + \cdots + \alpha_n v_n$. Find a necessary and sufficient condition (in terms of $\alpha_1, \ldots, \alpha_n$) for the linear independence of the vectors $v_1 + w, \ldots, v_n + w$.

Problem 5. Let D be a finite set and let $\mathbb{F}$ be a scalar field. Then the set of all functions defined on D with values in $\mathbb{F}$ is a vector space over $\mathbb{F}$ with the addition and scalar multiplication of functions defined pointwise. This space is denoted by $\mathbb{F}^D$.

(a) Prove that $\mathbb{F}^D$ is finite dimensional if and only if D is finite.

(b) If D is finite, then $\dim\bigl(\mathbb{F}^D\bigr) = |D|$.

Problem 6. Consider the vector space $\mathcal{P}_2$ (over the field of real numbers $\mathbb{R}$) of all polynomials with real coefficients of degree smaller or equal than 2. Let $s, t \in \mathbb{R}$. Consider the following two subsets of $\mathcal{P}_2$:

$$\mathcal{Z}_s := \{ p \in \mathcal{P}_2 : p(s) = 0 \} \quad \text{and} \quad \mathcal{V}_t := \{ p \in \mathcal{P}_2 : p'(t) = 0 \}.$$

(a) Prove that $\mathcal{Z}_s$ is a subspace of $\mathcal{P}_2$. Find a basis of this subspace. What is $\dim \mathcal{Z}_s$?

(b) Prove that $\mathcal{V}_t$ is a subspace of $\mathcal{P}_2$. Find a basis of this subspace. What is $\dim \mathcal{V}_t$?

(c) Let $s, t \in \mathbb{R}$, $s \neq t$. Describe the polynomials in each of the subspaces $\mathcal{Z}_s \cap \mathcal{Z}_t$, $\mathcal{V}_t \cap \mathcal{Z}_s$ and $\mathcal{V}_s \cap \mathcal{V}_t$. Find a basis for each of these subspaces.

(d) Let $s, t \in \mathbb{R}$ be given. Solve the equation $\mathcal{Z}_s \cap \mathcal{Z}_u = \mathcal{V}_v \cap \mathcal{Z}_t$ for u and v.