MATH 504 Assignment 2 February 18, 2020 Name ____________________

Problem 1. Let $\mathcal{V}$ and $\mathcal{W}$ be finite-dimensional vector spaces over a scalar field $\mathbb{F}$. Let $\mathcal{S}$ be a subspace of the direct product vector space $\mathcal{V}\times\mathcal{W}$. Define the following four subspaces

$$\begin{aligned}
\operatorname{dom}\mathcal{S} &= \bigl\{v \in \mathcal{V} : \exists\, w \in \mathcal{W} \text{ such that } (v,w) \in \mathcal{S}\bigr\},\\
\operatorname{ran}\mathcal{S} &= \bigl\{w \in \mathcal{W} : \exists\, v \in \mathcal{V} \text{ such that } (v,w) \in \mathcal{S}\bigr\},\\
\operatorname{nul}\mathcal{S} &= \bigl\{v \in \mathcal{V} : \ (v,0_{\mathcal{W}}) \in \mathcal{S}\bigr\},\\
\operatorname{mul}\mathcal{S} &= \bigl\{w \in \mathcal{W} : \ (0_{\mathcal{V}},w) \in \mathcal{S}\bigr\}.
\end{aligned}$$

Prove the equality

$$\dim\operatorname{dom}\mathcal{S} + \dim\operatorname{mul}\mathcal{S} = \dim\operatorname{ran}\mathcal{S} + \dim\operatorname{nul}\mathcal{S}.$$

Hint: The following equivalence holds. For all $v \in \mathcal{V}$ and all $w \in \mathcal{W}$ we have:

$$(v,w) \in \mathcal{S} \quad \Leftrightarrow \qquad \bigl(\forall x \in \operatorname{nul}\mathcal{S}\bigr) \wedge \bigl(\forall y \in \operatorname{mul}\mathcal{S}\bigr) \quad (v+x, w+y) \in \mathcal{S}.$$

Problem 2. Let $\mathcal{V}$ be a vector space over $\mathbb{F}$ and $T \in \mathcal{L}(\mathcal{V})$. Assume that there exists a function $f : \mathcal{V} \to \mathbb{F}$ such that $Tv = f(v)\, v$ for each $v \in \mathcal{V}$. Prove that T is a multiple of the identity operator. That is, there exists $\alpha \in \mathbb{F}$ such that $Tv = \alpha\, v$ for each $v \in \mathcal{V}$. (A plain English explanation: The equation $Tv = f(v)\, v$ is telling us that T scales each vector in $\mathcal{V}$ by the scaling coefficient $f(v)$. The point of the problem is to prove that T must scale each vector by the same coefficient. This is a consequence of the linearity of T.)

Problem 3. Let $\mathcal{V}$ be a nontrivial finite dimensional vector space, $n = \dim\ \mathcal{V}$, and let $T : \mathcal{V} \to \mathcal{V}$ be a linear operator. Define recursively:

$$T^0 = I \quad \text{and} \quad \forall\, j \in \mathbb{N} \quad T^j = T^{j-1} \circ T.$$

(a) Prove that there exists $k \in \{1,\dots,n\}$ such that $\operatorname{nul}(T^k) = \operatorname{nul}(T^{k+1})$.

(b) If $\operatorname{nul}(T) \neq \{0_{\mathcal{V}}\}$, then there exists $k \in \{1,\dots,n\}$ such that

$$\forall\, j \in \{1,\dots,k\} \quad \operatorname{nul}(T^{j-1}) \subsetneq \operatorname{nul}(T^j).$$

and

$$\forall\, l \in \mathbb{N} \setminus \{1,\dots,k\} \quad \operatorname{nul}(T^k) = \operatorname{nul}(T^l).$$

(c) Explore $\operatorname{ran}(T^j)$ with $j \in \mathbb{N}$ in the spirit of (a) and (b). Formulate your statements and prove them.

Problem 4. Let $\mathbb{C}[z]$ be the set of all polynomials with complex coefficients. For $n \in \mathbb{N}$ by $\mathbb{C}[z]_{<n}$ we denote the complex vector subspace of $\mathbb{C}[z]$ of all polynomials whose degree is less than n. (You do not need to prove this claim.) By $D : \mathbb{C}[z] \to \mathbb{C}[z]$ we denote the differentiation operator

$$\bigl(Df\bigr)(z) = f'(z), \quad f \in \mathbb{C}[z].$$

Let $\mathcal{Q}$ be a nontrivial finite dimensional subspace of $\mathbb{C}[z]$. Prove that $D\mathcal{Q} \subseteq \mathcal{Q}$ if and only if there exists $n \in \mathbb{N}$ such that $\mathcal{Q} = \mathbb{C}[z]_{<n}$.

Problem 5. Let $(\mathcal{V}, \langle\,\cdot\,,\cdot\,\rangle)$ be an inner product space over a scalar field $\mathbb{F}$ with a positive definite inner product $\langle\,\cdot\,,\cdot\,\rangle$. Let $\|\cdot\|$ be the corresponding norm on $\mathcal{V}$. That is, for $v \in \mathcal{V}$, $\|v\| := \sqrt{\langle v, v\rangle}$. Find a necessary and sufficient condition (in terms of the vectors $v_1,\dots,v_k \in \mathcal{V}$) for the following equality

$$\|v_1 + \cdots + v_k\| = \|v_1\| + \cdots + \|v_k\|.$$

Problem 6. Let $\mathcal{V}$ be a finite dimensional vector space over a scalar field $\mathbb{F}$. Assume that $\dim \mathcal{V} > 1$. Let $\langle \cdot, \cdot \rangle$ be a positive definite inner product on $\mathcal{V}$. Let x and y be fixed nonzero vectors in $\mathcal{V}$. Define the operator $T \in \mathcal{L}(\mathcal{V})$ by

$$Tv = v - \langle v, x \rangle y, \qquad v \in \mathcal{V}.$$

You do not need to prove that $T \in \mathcal{L}(\mathcal{V})$. Answer the following questions and provide complete rigorous justifications.

(a) Determine all eigenvalues and the corresponding eigenspaces of T. Provide a proof that you indeed found <u>all</u> the eigenvalues.

(b) Determine an explicit formula for T^*.

(c) Determine a necessary and sufficient condition for $Q \in \mathcal{L}(\mathcal{V})$ to commute with T.

(d) Determine a necessary and sufficient condition for T to be normal.

(e) Determine a necessary and sufficient condition for T to be self-adjoint.

Problem 7. Let $\mathcal{U}$ and $\mathcal{V}$ be finite-dimensional vector spaces over the complex field $\mathbb{C}$ such that $m = \dim U$ and $n = \dim V$ are positive integers. Assume that $\langle \cdot, \cdot \rangle_{\mathcal{U}}$ is a positive definite inner product on $\mathcal{U}$ and $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ is a positive definite inner product on $\mathcal{V}$. Let $T \in \mathcal{L}(\mathcal{U}, \mathcal{V})$.

(a) Prove that the operator T^*T is a nonnegative operator on $(\mathcal{U}, \langle \cdot, \cdot \rangle_{\mathcal{U}})$ and

$$\operatorname{nul}(T^*T) = \operatorname{nul} T.$$

(b) Define an operator $S : \operatorname{ran}(T^*) \to \operatorname{ran} T$ by

$$Su = Tu \quad \text{for all} \quad u \in \operatorname{ran}(T^*).$$

Prove that S is an isomorphism between the subspaces $\operatorname{ran}(T^*)$ of $\mathcal{U}$ and $\operatorname{ran} T$ of $\mathcal{V}$.

Problem 8. Let $\mathcal{U}$ and $\mathcal{V}$ be finite-dimensional vector spaces over the complex field $\mathbb{C}$ such that $m = \dim U$ and $n = \dim V$ are positive integers. Assume that $\langle \cdot, \cdot \rangle_{\mathcal{U}}$ is a positive definite inner product on $\mathcal{U}$ and $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ is a positive definite inner product on $\mathcal{V}$. Let $T \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ be such that $T \neq 0_{\mathcal{L}(\mathcal{U},\mathcal{V})}$.
Prove the existence of the following objects:

(i) an orthonormal basis $\mathcal{B} = \{u_1, \ldots, u_m\}$ for $\mathcal{U}$,

(ii) an orthonormal basis $\mathcal{C} = \{v_1, \ldots, v_n\}$ for $\mathcal{V}$,

(iii) a positive integer r such that $r \leq \min\{m, n\}$,

(iv) positive real numbers $\sigma_1, \ldots, \sigma_r$ such that $\sigma_1 \geq \cdots \geq \sigma_r$,

such that

$$M_{\mathcal{C}}^{\mathcal{B}}(T) = \left[\begin{array}{ccc|c} \sigma_1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & 0_{r \times (m-r)} \\ 0 & \cdots & \sigma_r & \\ \hline & 0_{(n-r) \times r} & & 0_{(n-r) \times (m-r)} \end{array}\right].$$