Theorem 1. *Let $\mathscr{V}$ be a finite-dimensional vector space over a scalar field $\mathbb{F}$ with $\dim \mathscr{V} = n \in \mathbb{N}$. Let $T \in \mathscr{L}(\mathscr{V})$ and assume that there exists a basis $\mathscr{B} = \bigl(v_1, \ldots, v_n\bigr)$ of $\mathscr{V}$ for which the matrix $M_{\mathscr{B}}^{\mathscr{B}}(T)$ is upper-triangular with diagonal entries a_{jj} where $j \in \{1, \ldots, n\}$. Then T is not injective if and only if there exists $i \in \{1, \ldots, n\}$ such that $a_{ii} = 0$.*

Proof. Let

$$M_{\mathscr{B}}^{\mathscr{B}}(T) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{jj} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & a_{nn} \end{bmatrix},$$

or, in English, the entries of the matrix $M_{\mathscr{B}}^{\mathscr{B}}(T)$ are $a_{kj} \in \mathbb{F}$ with $k, j \in \{1, \ldots, n\}$ and $a_{kj} = 0$ whenever $k > j$. By the definition of the matrix $M_{\mathscr{B}}^{\mathscr{B}}(T)$, this means that for every $j \in \{1, \ldots, n\}$ we have

$$T v_j = \sum_{k=1}^{j} a_{kj} v_k. \tag{1}$$

We first prove the "if" part of the claim. Assume that there exists $i \in \{1, \ldots, n\}$ such that $a_{ii} = 0$. Set

$$\mathscr{U} = \operatorname{span}\{v_1, \ldots, v_i\}.$$

By (1), for every $j \in \{1, \ldots, i\}$ we have

$$T v_j = \sum_{k=1}^{j} a_{kj} v_k = \sum_{k=1}^{i} a_{kj} v_k \in \mathscr{U}. \tag{2}$$

It follows from the preceding i equalities that for every $u \in \mathscr{U}$ we have $Tu \in \mathscr{U}$. Therefore, the restriction of T to $\mathscr{U}$, that is, the operator S defined by $Su = Tu$ for all $u \in \mathscr{U}$ is an operator in $\mathscr{L}(\mathscr{U})$.

Since $a_{ii} = 0$, the equalities in (2) read: for every $j \in \{1, \ldots, i\}$ we have

$$S v_j = T v_j = \sum_{k=1}^{j} a_{kj} v_k = \sum_{k=1}^{i-1} a_{kj} v_k \in \operatorname{span}\{v_1, \ldots, v_{i-1}\}.$$

Consequently, for every $u \in \mathscr{U}$ we have

$$Su = Tu \in \operatorname{span}\{v_1, \ldots, v_{i-1}\}.$$

Hence, $v_i \notin \operatorname{ran}(S)$. That is, $\operatorname{ran}(S) \subsetneq \mathscr{U}$, or equivalently $\dim \operatorname{ran}(S) < \dim \mathscr{U}$. By the Nullity-Rank theorem, $\dim \operatorname{nul}(S) = \dim \mathscr{U} - \dim \operatorname{ran}(S) \geq 1$. Thus, $\operatorname{nul}(S) \neq \{0_{\mathscr{V}}\}$. Let $u \in \mathscr{U} \subseteq \mathscr{V}$ be such that $u \neq 0_{\mathscr{V}}$ and $Su = 0_{\mathscr{V}}$. Since $Tu = Su = 0_{\mathscr{V}}$, it has been proven that T is not an injection.

Next we prove the "only if" part of the claim. Assume that T is not injective. It is convenient to introduce the following notation: for every $j \in \{1, \ldots, n\}$ set

$$\mathscr{U}_j = \operatorname{span}\{v_1, \ldots, v_j\}.$$

Notice that $\mathscr{U}_n = \mathscr{V}$ and, if $n > 1$, for all $j \in \{2, \ldots, n\}$ we have $\mathscr{U}_{j-1} \subsetneq \mathscr{U}_j$p. Since the vectors $v_1, \ldots, v_n$ are linearly independent, for all $j \in \{1, \ldots, n\}$ we have

$$\dim \mathscr{U}_j = j. \tag{3}$$

The equalities in (1) imply that for every $j \in \{1, \dots, n\}$ we have

$$T\mathscr{U}_j \subseteq \mathscr{U}_j. \tag{4}$$

Since T is not injective, we have $\operatorname{nul}(T) \neq \{0_{\mathscr{V}}\}$, that is $\dim \operatorname{nul}(T) \geq 1$. By the Nullity-Rank theorem, $\dim \operatorname{ran}(T) = n - \dim \operatorname{nul}(T) < n$. Consequently, $\operatorname{ran}(T) = T\mathscr{V} \subsetneq \mathscr{V}$. Since $\mathscr{U}_n = \mathscr{V}$, we also have $T\mathscr{U}_n \subsetneq \mathscr{U}_n$.

Consider the set

$$\mathbb{K} = \bigl\{ j \in \{1, ..., n\} \, : \, T\mathscr{U}_j \subsetneq \mathscr{U}_j \bigr\}.$$

Since $T\mathscr{U}_n \subsetneq \mathscr{U}_n$, we have $n \in \mathbb{K}$. Hence, the set $\mathbb{K}$ is a nonempty set of positive integers. By the Well-Ordering Axiom of Integers $\min \mathbb{K}$ exists. Set $m = \min \mathbb{K}$.

Case 1. $m = 1$. In this case $T\mathscr{U}_1 \subsetneq \mathscr{U}_1$. Consequently, $\dim(T\mathscr{U}_1) < \dim(\mathscr{U}_1)$. Since $\dim \mathscr{U}_1 = 1$, we deduce that $\dim(T\mathscr{U}_1) = 0$. Thus $T\mathscr{U}_1 = \{0_{\mathscr{V}}\}$, so $Tv_1 = 0_{\mathscr{V}}$. Hence $C_{\mathscr{B}}(Tv_1) = [0 \, \cdots \, 0]^\top$ and so $a_{11} = 0$.

Case 2. $m \in \{2, \dots, n\}$. Then $m - 1 \in \{1, \dots, n\}$. By the definition of minimum, we have that $m - 1 \notin \mathbb{K}$. Consequently,

$$T\mathscr{U}_{m-1} \subsetneq \mathscr{U}_{m-1} \quad \text{is not true.}$$

By (4), we have $T\mathscr{U}_{m-1} \subseteq \mathscr{U}_{m-1}$. The last inclusion is equivalent to

$$T\mathscr{U}_{m-1} \subsetneq \mathscr{U}_{m-1} \quad \vee \quad T\mathscr{U}_{m-1} = \mathscr{U}_{m-1}.$$

Since we proved that $T\mathscr{U}_{m-1} \subsetneq \mathscr{U}_{m-1}$ is not true, we must have $T\mathscr{U}_{m-1} = \mathscr{U}_{m-1}$. (This logical reasoning $(p \vee q) \wedge (\neg q) \Rightarrow p$ is called "disjunctive syllogism.")

Since $m \in \mathbb{K}$ we have

$$T\mathscr{U}_m \subsetneq \mathscr{U}_m.$$

Further, by definition of $\mathscr{U}_{m-1}$ and $\mathscr{U}_m$, we have $\mathscr{U}_{m-1} \subsetneq \mathscr{U}_m$. Hence $T\mathscr{U}_{m-1} \subset T\mathscr{U}_m$.

Now we collect all the information that we have about $\mathscr{U}_{m-1}$, $T\mathscr{U}_{m-1}$, $\mathscr{U}_m$, $T\mathscr{U}_m$:

$$\mathscr{U}_{m-1} = T\mathscr{U}_{m-1} \subseteq T\mathscr{U}_m \subsetneq \mathscr{U}_m.$$

Using (3), for the corresponding dimensions we deduce

$$m - 1 = \dim(\mathscr{U}_{m-1}) \leq \dim(T\mathscr{U}_m) < \dim(\mathscr{U}_m) = m.$$

Since $\dim(T\mathscr{U}_m)$ is a positive integer, the preceding relation among positive integers yields

$$m - 1 = \dim(T\mathscr{U}_m).$$

Since

$$\mathscr{U}_{m-1} \subseteq T\mathscr{U}_m \quad \text{and} \quad m - 1 = \dim(\mathscr{U}_{m-1}) \quad \text{and} \quad m - 1 = \dim(T\mathscr{U}_m),$$

we deduce

$$T\mathscr{U}_m = \mathscr{U}_{m-1}.$$

Since by the definition of $\mathscr{U}_m$ we have $v_m \in \mathscr{U}_m$, the preceding set equality yields

$$Tv_m \in \mathscr{U}_{m-1} = \operatorname{span}\{v_1, \dots, v_{m-1}\}.$$

Thus, there exist $\alpha_1, \dots, \alpha_{m-1} \in \mathbb{F}$ such that

$$Tv_m = \alpha_1 v_1 + \cdots + \alpha_{m-1} v_{m-1}.$$

By (1), that is by the definition of $M_{\mathscr{B}}^{\mathscr{B}}(T)$ we have,

$$Tv_m = \sum_{k=1}^{m} a_{km} v_k.$$

Since the vectors $v_1, \dots, v_m$ are linearly independent, the last two equalities imply that $a_{mm} = 0$. $\square$

Theorem 2 (5.41 page 157 in the textbook)**.** *Let $\mathscr{V}$ be a finite-dimensional vector space over a scalar field $\mathbb{F}$ with $\dim \mathscr{V} = n \in \mathbb{N}$. Let $T \in \mathscr{L}(\mathscr{V})$ and assume that there exists a basis $\mathscr{B}$ of $\mathscr{V}$ for which the matrix $M_{\mathscr{B}}^{\mathscr{B}}(T)$ is upper-triangular with diagonal entries a_{jj} where $j \in \{1, \ldots, n\}$. Then*

$$\sigma(T) = \bigl\{a_{jj} : j \in \{1, ..., n\}\bigr\}.$$

Proof. We proved before that $M_{\mathscr{B}}^{\mathscr{B}} : \mathscr{L}(\mathscr{V}) \to \mathbb{F}^{n\times n}$ is an isomorphism of algebras. Therefore

$$M_{\mathscr{B}}^{\mathscr{B}}(T - \lambda I) = M_{\mathscr{B}}^{\mathscr{B}}(T) - \lambda M_{\mathscr{B}}^{\mathscr{B}}(I) = M_{\mathscr{B}}^{\mathscr{B}}(T) - \lambda I_n.$$

Here I_n denotes the identity matrix in $\mathbb{F}^{n\times n}$. As $M_{\mathscr{B}}^{\mathscr{B}}(T)$ and $M_{\mathscr{B}}^{\mathscr{B}}(I) = I_n$ are upper triangular, $M_{\mathscr{B}}^{\mathscr{B}}(T - \lambda I)$ is upper triangular as well. Its diagonal entries are $a_{jj} - \lambda$, where $j \in \{1, ..., n\}$.

To prove the set equality

$$\sigma(T) = \bigl\{a_{jj} : j \in \{1, ..., n\}\bigr\}.$$

in the theorem we need to prove two inclusions.

First we prove $\subseteq$. Let $\lambda \in \sigma(T)$. Because λ is an eigenvalue, $T - \lambda I$ is not injective. Because $T - \lambda I$ is not injective. By Theorem 1 one of the diagonal entries of the upper triangular matrix

$$M_{\mathscr{B}}^{\mathscr{B}}(T - \lambda I) = M_{\mathscr{B}}^{\mathscr{B}}(T) - \lambda I_n$$

is zero. That is, there exists $i \in \{1, ..., n\}$ such that $a_{ii} - \lambda = 0$. Thus $\lambda = a_{ii}$, and we proved

$$\sigma(T) \subseteq \bigl\{a_{jj} : j \in \{1, ..., n\}\bigr\}.$$

Next we prove $\supseteq$. Let $j \in \{1, ..., n\}$ be arbitrary. Consider the matrix $M_{\mathscr{B}}^{\mathscr{B}}(T - a_{jj} I)$. The j-th diagonal entry of the matrix

$$M_{\mathscr{B}}^{\mathscr{B}}(T - a_{jj} I) = M_{\mathscr{B}}^{\mathscr{B}}(T) - a_{jj} I_n$$

is equal to $a_{jj} - a_{jj} = 0$. By Theorem 1 the operator $T - a_{jj} I$ is not injective. This implies that a_{jj} is an eigenvalue of T. Thus $a_{jj} \in \sigma(T)$. This completes the proof. $\square$