

VECTOR SPACES

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In these notes we denote \mathbb{C} denotes the set of all complex numbers, \mathbb{R} denotes the set of all real numbers, \mathbb{Z} denotes the set of all integers and \mathbb{N} denotes the set of all positive integers.

1. AXIOMS

Definition 1.1. A subset \mathbb{F} of \mathbb{C} is called a *scalar field* if the following five statements hold.

SF1 $0, 1 \in \mathbb{F}$.

SF2 For all $\alpha, \beta \in \mathbb{F}$ we have $\alpha + \beta \in \mathbb{F}$.

SF3 For all $\alpha, \beta \in \mathbb{F}$ we have $\alpha\beta \in \mathbb{F}$.

SF4 For all $\alpha \in \mathbb{F}$ we have $-\alpha \in \mathbb{F}$.

SF5 For all $\alpha \in \mathbb{F} \setminus \{0\}$ we have $\frac{1}{\alpha} \in \mathbb{F}$.

SF6 For all $\alpha \in \mathbb{F}$ we have $\bar{\alpha} \in \mathbb{F}$.

Proposition 1.2. If \mathbb{F} is a scalar field, then $\mathbb{Q} \subseteq \mathbb{F}$.

Proof. Hint: First use Mathematical Induction to prove $\mathbb{N} \subset \mathbb{F}$. Second, use the fact that $\alpha \in \mathbb{Z}$ if and only if $\alpha = 0$ or $\alpha \in \mathbb{N}$ or $-\alpha \in \mathbb{N}$ to prove that $\mathbb{Z} \subset \mathbb{F}$. Finally, prove that for arbitrary $\alpha \in \mathbb{Z}$ and arbitrary $\beta \in \mathbb{N}$ we have $\alpha/\beta \in \mathbb{F}$. \square

Definition 1.3. Let \mathcal{V} be a nonempty set and let \mathbb{F} be a scalar field. The set \mathcal{V} is called a *vector space over \mathbb{F}* if the following ten conditions are satisfied.

AE There exists a function $+$: $\mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$.

(The mapping in **AE** is called *addition* and its value on a pair $(u, v) \in \mathcal{V} \times \mathcal{V}$ is denoted by $u + v$.)

AA For all $u, v, w \in \mathcal{V}$ we have $u + (v + w) = (u + v) + w$.

AC For all $u, v \in \mathcal{V}$ we have $u + v = v + u$.

AZ There exists an element $0_{\mathcal{V}} \in \mathcal{V}$ such that $v + 0_{\mathcal{V}} = v$ for all $v \in \mathcal{V}$.

AO For each $v \in \mathcal{V}$ there exists $w \in \mathcal{V}$ such that $v + w = 0_{\mathcal{V}}$.

SE There exists a function \cdot : $\mathbb{F} \times \mathcal{V} \rightarrow \mathcal{V}$.

(The mapping in **SE** is called *scaling* and its value on a pair $(\alpha, v) \in \mathbb{F} \times \mathcal{V}$ is denoted by $\alpha \cdot v$, or simply αv .)

SA For all $\alpha, \beta \in \mathbb{F}$ and all $v \in \mathcal{V}$ we have $\alpha(\beta v) = (\alpha\beta)v$.

SD For all $\alpha \in \mathbb{F}$ and all $u, v \in \mathcal{V}$ we have $\alpha(u + v) = \alpha u + \alpha v$.

- SD** For all $\alpha, \beta \in \mathbb{F}$ and all $v \in \mathcal{V}$ we have $(\alpha + \beta)v = \alpha v + \beta v$.
SO For all $v \in \mathcal{V}$ we have $1v = v$.

2. BASIC PROPOSITIONS

Few immediate consequences of Definitions 1.3 and 1.1 are collected in the following propositions.

Proposition 2.1. *Let \mathcal{V} be a vector space over a scalar field \mathbb{F} . Then for every $v \in \mathcal{V}$ we have $0v = 0_{\mathcal{V}}$.*

Proof. Let $v \in \mathcal{V}$ be arbitrary. Then by **SE** we have that $0v \in \mathcal{V}$. By **AO** there exists $w \in \mathcal{V}$ such that $0v + w = 0_{\mathcal{V}}$. Then

$$\begin{aligned}
 0_{\mathcal{V}} &= 0v + w && \text{by the choice of } w \\
 &= (0 + 0)v + w && \text{since } 0 + 0 = 0 \text{ in } \mathbb{C}, \text{ **SE** and **AE**} \\
 &= (0v + 0v) + w && \text{by **SD**, **AE**} \\
 &= 0v + (0v + w) && \text{by **AA**} \\
 &= 0v + 0_{\mathcal{V}} && \text{by the choice of } w, 0v + w = 0_{\mathcal{V}}, \text{ **AE**} \\
 &= 0v && \text{by **AZ**.}
 \end{aligned}$$

The presented sequence of equalities proves the proposition. \square

The proof of the next proposition is similar.

Proposition 2.2. *Let \mathcal{V} be a vector space over a scalar field \mathbb{F} . Then for every $\alpha \in \mathbb{F}$ we have $\alpha 0_{\mathcal{V}} = 0_{\mathcal{V}}$.*

Proof. Let $\alpha \in \mathbb{F}$ be arbitrary. Then by **SE** we have that $\alpha 0_{\mathcal{V}} \in \mathcal{V}$. By **AO** there exists $w \in \mathcal{V}$ such that $\alpha 0_{\mathcal{V}} + w = 0_{\mathcal{V}}$. Then

$$\begin{aligned}
 0_{\mathcal{V}} &= \alpha 0_{\mathcal{V}} + w && \text{by the choice of } w \\
 &= \alpha(0_{\mathcal{V}} + 0_{\mathcal{V}}) + w && \text{by **AZ**, **SE** and **AE**} \\
 &= (\alpha 0_{\mathcal{V}} + \alpha 0_{\mathcal{V}}) + w && \text{by **SD**, and **AE**} \\
 &= \alpha 0_{\mathcal{V}} + (\alpha 0_{\mathcal{V}} + w) && \text{by **AA**} \\
 &= \alpha 0_{\mathcal{V}} + 0_{\mathcal{V}} && \text{by the choice of } w \text{ and **AE**} \\
 &= \alpha 0_{\mathcal{V}} && \text{by **AZ**.}
 \end{aligned}$$

The presented sequence of equalities proves the proposition. \square

Proposition 2.3. *Let \mathcal{V} be a vector space over a scalar field \mathbb{F} . For every $v \in \mathcal{V}$ the following equivalence holds*

$$v + w = 0_{\mathcal{V}} \Leftrightarrow w = (-1)v. \quad (2.1)$$

Proof. Let $v \in \mathcal{V}$ be arbitrary. First we will prove \Leftarrow in (2.1). Let $w = (-1)v$. Then

$$\begin{aligned} v + w &= v + (-1)v && \text{by } \mathbf{AE} \\ &= 1v + (-1)v && \text{by } \mathbf{SO}, \text{ and } \mathbf{AE} \\ &= (1 + (-1))v && \text{by } \mathbf{SD} \\ &= 0v && \text{by } 1 + (-1) = 0 \text{ in } \mathbb{C} \text{ and } \mathbf{SE} \\ &= 0_{\mathcal{V}} && \text{by Proposition 2.1} \end{aligned}$$

The presented sequence of equalities proves \Leftarrow in (2.1).

Next we prove the converse, that is \Rightarrow in (2.1). Assume $v + w = 0_{\mathcal{V}}$. Then

$$\begin{aligned} w &= 0_{\mathcal{V}} + w && \text{by } \mathbf{AZ} \text{ and } \mathbf{AC} \\ &= 0v + w && \text{by Proposition 2.1 and } \mathbf{AE} \\ &= ((-1) + 1)v + w && \text{by } (-1) + 1 = 0 \text{ in } \mathbb{C}, \mathbf{SE} \text{ and } \mathbf{AE} \\ &= ((-1)v + 1v) + w && \text{by } \mathbf{SD} \text{ and } \mathbf{AE} \\ &= (-1)v + (v + w) && \text{by } \mathbf{AA}, \mathbf{SO} \text{ and } \mathbf{AE} \\ &= (-1)v + 0_{\mathcal{V}} && \text{by } v + w = 0_{\mathcal{V}}, \text{ and } \mathbf{AE} \\ &= (-1)v && \text{by } \mathbf{AZ} \end{aligned}$$

The presented sequence of equalities proves \Rightarrow in (2.1).

Since $v \in \mathcal{V}$ was arbitrary, the proposition is proved. \square

Definition 2.4. Let \mathcal{V} be a vector space over a scalar field \mathbb{F} and let $v \in \mathcal{V}$. The unique solution of equation $v + x = 0_{\mathcal{V}}$ is denoted by $-v$ and it is called the *opposite* of v .

Definition 2.5. Let \mathcal{V} be a vector space over a scalar field \mathbb{F} , let $n \in \mathbb{N} \setminus \{1\}$, and for every $k \in \{1, \dots, n\}$ let $v_k \in \mathcal{V}$. The sum

$$\sum_{k=1}^n v_k$$

is defined by the finite recursion:

$$\sum_{k=1}^1 v_k = v_1, \quad \forall m \in \{2, \dots, n\} \quad \text{we set} \quad \sum_{k=1}^m v_k = \left(\sum_{k=1}^{m-1} v_k \right) + v_m$$

For example, if $v_1, v_2, v_3, v_4, v_5 \in \mathcal{V}$, then

$$v_1 + v_2 + v_3 + v_4 + v_5 = \left(\left((v_1 + v_2) + v_3 \right) + v_4 \right) + v_5$$

3. EXAMPLES

Example 3.1. Let \mathbb{F} be a scalar field. Then $\mathcal{V} = \mathbb{F}$ is a vector space over \mathbb{F} . The addition in $\mathcal{V} = \mathbb{F}$ is the addition of complex numbers in \mathbb{F} and the scaling in $\mathcal{V} = \mathbb{F}$ is just the multiplication of complex numbers. The axioms

of the vector space then follow from the axioms of the scalar field and the properties of the complex numbers.

The next example is a generalization of the previous one.

Example 3.2. Let \mathbb{F} and \mathbb{K} be scalar fields such that $\mathbb{F} \subseteq \mathbb{K}$. Then $\mathcal{V} = \mathbb{K}$ is a vector space over \mathbb{F} . The addition in $\mathcal{V} = \mathbb{K}$ is the addition of complex numbers in \mathbb{K} and the scaling in $\mathcal{V} = \mathbb{K}$ is just the multiplication of complex numbers. The axioms of the vector space then follow from the axioms of the scalar field and the properties of the complex numbers.

Example 3.3. This is the quintessential example of a vector space. Many other vector spaces are special cases of this example. Let D be an arbitrary nonempty set and let \mathbb{F} be a scalar field. Let \mathcal{V} be the set of all functions from D to \mathbb{F} . This set is denoted by \mathbb{F}^D . The addition in \mathbb{F}^D is defined as follows: let $f, g \in \mathbb{F}^D$, the function $f + g$ is defined by

$$(f + g)(t) := f(t) + g(t) \quad \text{for all } t \in D.$$

The scaling in \mathbb{F}^D is defined as follows: let $\alpha \in \mathbb{F}$ and $f \in \mathbb{F}^D$, the function αf is defined by

$$(\alpha f)(t) := \alpha f(t) \quad \text{for all } t \in D.$$

The above definitions of addition and scaling of functions are called *pointwise* definitions. As an exercise you should go through the proofs of all the axioms of the vector space for this specific case.

Example 3.4. This is a special case of Example 3.3. Let $n \in \mathbb{N}$ and

$$D = \{t \in \mathbb{N} : t \leq n\} = \{1, \dots, n\}.$$

Sometimes this set is written simply as $D = \{1, \dots, n\}$. Then the vector space \mathbb{F}^D can be identified with the space \mathbb{F}^n of all n -tuples of elements of \mathbb{F} .

Example 3.5. This is another special case of Example 3.3. Let $m, n \in \mathbb{N}$ and

$$D = \{(s, t) \in \mathbb{N} \times \mathbb{N} : s \leq m \wedge t \leq n\};$$

that is $D = \{1, \dots, m\} \times \{1, \dots, n\}$. Then \mathbb{F}^D can be identified with the space $\mathbb{F}^{m \times n}$ of all $m \times n$ matrices with entries in \mathbb{F} .

Example 3.6. Let \mathbb{F} be a scalar field. By $\mathbb{F}[z]$ we denote the set of all polynomials in variable z with coefficients from the scalar field \mathbb{F} . Then $\mathbb{F}[z]$ is a vector space with addition and scalar multiplication defined pointwise.

The next example is a generalization of Example 3.3,

Example 3.7. Let D be an arbitrary nonempty set and let \mathcal{V} be a vector space over a scalar field \mathbb{F} . Let \mathcal{W} be the set of all functions from D to \mathcal{V} ; that is $\mathcal{W} = \mathcal{V}^D$. With the addition and scaling of functions defined pointwise, \mathcal{W} is a vector space over \mathbb{F} . The functions in \mathcal{V}^D are said to be *vector valued* functions.

4. SET OPERATIONS IN A VECTOR SPACE

In a set theory class we learned about set operations. For two sets A and B we defined $A \cap B$, $A \cup B$, $A \setminus B$ and $A \Delta B$. In a vector space \mathcal{V} over a field \mathbb{F} fun with subsets is enriched by two more set operations: the addition of sets and scaling of sets.

Definition 4.1. Let \mathcal{V} be a vector space over a scalar field \mathbb{F} and let \mathcal{A} and \mathcal{B} be nonempty subsets of \mathcal{V} . We define the sum of $\mathcal{A} + \mathcal{B}$ by

$$\mathcal{A} + \mathcal{B} = \{u + v : u \in \mathcal{A}, v \in \mathcal{B}\}.$$

For $\alpha \in \mathbb{F}$ we define $\alpha\mathcal{A}$ by

$$\alpha\mathcal{A} = \{\alpha u : u \in \mathcal{A}\}.$$

Let $n \in \mathbb{N}$ and let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be subsets of \mathcal{V} . By recursion we define

$$\mathcal{A}_1 + \dots + \mathcal{A}_k := (\mathcal{A}_1 + \dots + \mathcal{A}_{k-1}) + \mathcal{A}_k, \quad k = 2, \dots, n.$$

By **AA**, the set $\mathcal{A}_1 + \dots + \mathcal{A}_n$ consists of all the sums $v_1 + \dots + v_n$ where $v_j \in \mathcal{A}_j$ where $j \in \{1, \dots, n\}$.

5. SPECIAL SUBSETS OF A VECTOR SPACE

The following definition distinguishes important subsets of a vector space \mathcal{V} over a field \mathbb{F} .

Definition 5.1. Let \mathcal{V} be a vector space over a scalar field \mathbb{F} . A subset \mathcal{U} of \mathcal{V} is said to be a *subspace* of \mathcal{V} if the following three conditions are satisfied:

- SuZ** $0_{\mathcal{V}} \in \mathcal{U}$.
- SuA** $\mathcal{U} + \mathcal{U} \subseteq \mathcal{U}$.
- SuS** For every $\alpha \in \mathbb{F}$ we have $\alpha\mathcal{U} \subseteq \mathcal{U}$

Proposition 5.2. *An intersection of subspaces of a vector space is also a subspace.*

Proposition 5.3. *A sum of subspaces of a vector space is also a subspace.*

A union of subspaces of a vector space is not necessarily a vector space. Problems 7.6 and 7.8 deal with this question.

Definition 5.4. Let \mathcal{V} be a vector space over \mathbb{R} . A nonempty subset \mathcal{C} of \mathcal{V} is said to be a *cone* in \mathcal{V} if $\alpha\mathcal{C} \subseteq \mathcal{C}$ for all $\alpha > 0$.

Definition 5.5. Let \mathcal{V} be a vector space over \mathbb{R} . A nonempty subset \mathcal{S} of \mathcal{V} is said to be a *convex set* in \mathcal{V} if $\alpha u + (1 - \alpha)v \in \mathcal{S}$ for all $\alpha \in [0, 1]$.

Exercise 5.6. Let \mathcal{V} be a vector space over \mathbb{R} and let \mathcal{C} be a cone in \mathcal{V} . Prove that \mathcal{C} is a convex set if and only if $\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$.

6. DIRECT SUMS OF SUBSPACES

Let \mathcal{V} be a vector space over a scalar field \mathbb{F} . Let \mathcal{U} and \mathcal{W} be subspaces of \mathcal{V} . Recall that $v \in \mathcal{U} + \mathcal{V}$ if and only if there exist $u \in \mathcal{U}$ and $w \in \mathcal{W}$ such that $v = u + w$. A stronger version of the last statement is in the following definition.

Definition 6.1. Let \mathcal{V} be a vector space over a scalar field \mathbb{F} and let \mathcal{U} and \mathcal{W} be subspaces of \mathcal{V} . The sum $\mathcal{U} + \mathcal{V}$ is called a *direct sum* if for every $v \in \mathcal{U} + \mathcal{V}$ there exist unique $u \in \mathcal{U}$ and $w \in \mathcal{W}$ such that $v = u + w$. The direct sum is denoted by $\mathcal{U} \oplus \mathcal{V}$.

For example, let $\mathbb{F} = \mathbb{R}$, $\mathcal{V} = \mathbb{R}^4$,

$$\mathcal{U} = \{(s_1, s_2, s_3, 0) : s_1, s_2, s_3 \in \mathbb{R}\} \text{ and } \mathcal{W} = \{(0, t_1, t_2, t_3) : t_1, t_2, t_3 \in \mathbb{R}\}.$$

Then $\mathbb{R}^4 = \mathcal{U} + \mathcal{W}$. However, this sum is not a direct sum. For $v = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ we can take $u = (x_1, s_2, s_3, 0) \in \mathcal{U}$ and $w = (0, x_2 - s_2, x_3 - s_3, x_4) \in \mathcal{W}$ with $s_2, s_3 \in \mathbb{R}$ arbitrary.

Setting

$$\mathcal{U} = \{(s_1, s_2, s_2, 0) : s_1, s_2 \in \mathbb{R}\} \quad \text{and} \quad \mathcal{W} = \{(0, -t_1, t_1, t_2) : t_1, t_2 \in \mathbb{R}\},$$

we have $\mathbb{R}^4 = \mathcal{U} \oplus \mathcal{W}$. Prove this as an exercise.

Proposition 6.2. Let \mathcal{V} be a vector space over a scalar field \mathbb{F} and let \mathcal{U} and \mathcal{W} be subspaces of \mathcal{V} . The following statements are equivalent:

- (a) The sum $\mathcal{U} + \mathcal{W}$ is direct.
- (b) For all $u, w \in \mathcal{V}$ we have

$$u \in \mathcal{U} \wedge w \in \mathcal{W} \wedge u + w = 0_{\mathcal{V}} \quad \Rightarrow \quad u = w = 0_{\mathcal{V}}. \quad (6.1)$$

- (c) $\mathcal{U} \cap \mathcal{W} = \{0_{\mathcal{V}}\}$.

Proof. The definition of the direct sum of $\mathcal{U} + \mathcal{W}$ written as an implication reads as follows: For all $v, u_1, w_1, u_2, w_2 \in \mathcal{V}$ we have

$$\forall j \in \{1, 2\} \ u_j \in \mathcal{U} \wedge w_j \in \mathcal{W} \wedge v = u_j + w_j \quad \Rightarrow \quad u_1 = u_2 \wedge w_1 = w_2. \quad (6.2)$$

Setting (6.1) $v = 0_{\mathcal{V}}$, $u_1 = u$, $w_1 = w$, $u_2 = w_2 = 0_{\mathcal{V}}$ in (6.2) we obtain the implication in (6.1). Therefore (a) implies (b).

Assume (b). Let $v \in \mathcal{U} \cap \mathcal{W}$ be arbitrary. Since $\mathcal{U} \cap \mathcal{W}$ is a subspace, $-v \in \mathcal{U} \cap \mathcal{W}$. Set $u = v$, $w = -v$ in (b). Then (b) implies $v = 0_{\mathcal{V}}$. This proves (c).

Assume (c). We need to prove the implication in (6.2). Let

$$v, u_1, w_1, u_2, w_2 \in \mathcal{V}$$

be arbitrary such that

$$\forall j \in \{1, 2\} \ u_j \in \mathcal{U} \wedge w_j \in \mathcal{W} \wedge v = u_j + w_j. \quad (6.3)$$

It follows from (6.3) that

$$0_{\mathcal{V}} = (u_1 + w_1) - (u_2 + w_2) = (u_1 - u_2) - (w_2 - w_1).$$

Consequently,

$$u_1 - u_2 = w_2 - w_1.$$

Since by (6.3) $u_1 - u_2 \in \mathcal{U}$ and $w_2 - w_1 \in \mathcal{W}$, we deduce from the last equality that

$$u_1 - u_2 = w_2 - w_1 \in \mathcal{U} \cap \mathcal{W} = \{0_{\mathcal{V}}\}.$$

Therefore, $u_1 = u_2$ and $w_1 = w_2$. This proves the implication in (6.2), proving (a).

Since we proved (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a), the propositions is proved. \square

Definition 6.3. Let \mathcal{V} be a vector space over a scalar field \mathbb{F} , let $n \in \mathbb{N}$ and let $\mathcal{U}_1, \dots, \mathcal{U}_n$ be subspaces of \mathcal{V} . The sum $\mathcal{U}_1 + \dots + \mathcal{U}_n$ is called a *direct sum* if for every $v \in \mathcal{U}_1 + \dots + \mathcal{U}_n$ there exist unique $u_j \in \mathcal{U}_j$, $j \in \{1, \dots, n\}$, such that $v = u_1 + \dots + u_n$. The direct sum is denoted by $\mathcal{U}_1 \oplus \dots \oplus \mathcal{U}_n$.

The preceding definition of the direct sum of subspaces written as an implication is as follows: The sum $\mathcal{U}_1 + \dots + \mathcal{U}_n$ is a direct sum if for all

$$v, u_1, \dots, u_n, w_1, \dots, w_n \in \mathcal{V}$$

the following implication holds

$$\begin{aligned} v = \sum_{k=1}^n u_k \wedge v = \sum_{k=1}^n w_k \wedge \forall k \in \{1, \dots, n\} u_k, w_k \in \mathcal{U}_k \\ \Rightarrow \forall k \in \{1, \dots, n\} u_k = w_k. \end{aligned} \quad (6.4)$$

Proposition 6.4. Let \mathcal{V} be a vector space over a scalar field \mathbb{F} , let $n \in \mathbb{N}$ and let $\mathcal{U}_1, \dots, \mathcal{U}_n$ be subspaces of \mathcal{V} . The following statements are equivalent:

- (a) The sum $\mathcal{U}_1 + \dots + \mathcal{U}_n$ is direct.
- (b) For all $u_1, \dots, u_n \in \mathcal{V}$ the following implication holds

$$\begin{aligned} 0_{\mathcal{V}} = \sum_{k=1}^n u_k \wedge \forall k \in \{1, \dots, n\} u_k \in \mathcal{U}_k \\ \Rightarrow \forall k \in \{1, \dots, n\} u_k = 0_{\mathcal{V}}. \end{aligned} \quad (6.5)$$

Proof. Assume (a). That is, assume that the implication in (6.4) holds. Setting $v = 0_{\mathcal{V}}$ and $w_k = 0_{\mathcal{V}}$ for all $k \in \{1, \dots, n\}$ in (6.4), the implication in (6.4) becomes (6.5). This proves (a) \Rightarrow (b).

Assume (b). Let

$$v, u_1, \dots, u_n, w_1, \dots, w_n \in \mathcal{V}$$

be arbitrary such that

$$v = \sum_{k=1}^n u_k \wedge v = \sum_{k=1}^n w_k \wedge \forall k \in \{1, \dots, n\} u_k, w_k \in \mathcal{U}_k. \quad (6.6)$$

The assumptions in (6.6) imply

$$0_{\mathcal{V}} = \sum_{k=1}^n (u_k - w_k) \wedge \forall k \in \{1, \dots, n\} u_k - w_k \in \mathcal{U}_k.$$

Now, by (6.5) we deduce

$$\forall k \in \{1, \dots, n\} \quad u_k - w_k = 0_{\mathcal{V}}.$$

This proves the implication in (6.4), proving (b).

The proposition is proved. \square

In the next theorem we prove that the Cartesian product of two vector spaces with appropriately defined vector addition and scalar multiplication is a vector space.

Theorem 6.5. *Let \mathcal{V} and \mathcal{X} be a vector spaces over a scalar field \mathbb{F} . Define the vector addition and scalar multiplication on the Cartesian product $\mathcal{V} \times \mathcal{X}$ as follows. For all $v, w \in \mathcal{V}$, all $x, y \in \mathcal{X}$ and all $\alpha \in \mathbb{F}$ set*

$$(v, x) + (w, y) = (v + w, x + y), \quad \alpha(v, x) = (\alpha v, \alpha x). \quad (6.7)$$

The set $\mathcal{V} \times \mathcal{X}$ with these two operations is a vector space.

Remark 6.6. Notice that the first plus sign in (6.7) is the addition in $\mathcal{V} \times \mathcal{X}$ which is being defined, the second plus sign is the addition in \mathcal{V} and the third plus sign is the addition in \mathcal{X} .

Definition 6.7. The set $\mathcal{V} \times \mathcal{X}$ with the operations defined in (6.7) is called the *direct product* of the vector spaces \mathcal{V} and \mathcal{X} .

7. PROBLEMS

Problem 7.1. In Definition 1.3 we use the same symbol $+$ to denote to different additions; one addition is the addition of complex numbers in a scalar field \mathbb{F} , the other addition is the addition of vectors in the vector space \mathcal{V} . Similarly, the usage of the blank space between two symbols is ambiguous; between two complex numbers it means the product of two complex numbers, while between a complex number and a vector in \mathcal{V} means scaling of that vector by a complex number. As a learner you should pay attention and make sure that you understand the meaning of the formulas that you are dealing with.

Let us introduce some “funny” names for the algebraic operations that appear in Definition 1.3.

$$\begin{array}{ll} \text{VectorPlus} : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}, & \text{Scale} : \mathbb{F} \times \mathcal{V} \rightarrow \mathcal{V} \\ \text{Plus} : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}, & \text{Times} : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}. \end{array}$$

Thus, for $u, v \in \mathcal{V}$ the sum of the vectors u and v is denoted by $\text{VectorPlus}(u, v)$, for $\alpha \in \mathbb{F}$ and $v \in \mathcal{V}$ the scaling of the vector v by α is denoted by $\text{Scale}(\alpha, v)$, for $\alpha, \beta \in \mathbb{F}$ the sum of the complex numbers α and β is denoted by $\text{Plus}(\alpha, \beta)$, and for $\alpha, \beta \in \mathbb{F}$ the product of the complex numbers α and β is denoted by $\text{Times}(\alpha, \beta)$.

Just to clarify, in this notation we have $\text{Plus}(2, 3) = 5$ and $\text{Times}(2, 3) = 6$. The distributive law for complex numbers in this notation reads: for all

complex numbers α, β and γ we have

$$\text{Times}(\alpha, \text{Plus}(\beta, \gamma)) = \text{Plus}(\text{Times}(\alpha, \beta), \text{Times}(\alpha, \gamma))$$

Finally, your task in this problem is to rewrite the axioms **SA**, **SD**, **SD**, and **SO** using the notation for the algebraic operations introduced above.

Problem 7.2. Let \mathbb{R}_+ denote the set of positive real numbers, set

$$\mathcal{V} = \mathbb{R}_+ \times \mathbb{R}_+ = (\mathbb{R}_+)^2,$$

and let $\mathbb{F} = \mathbb{R}$. Define the addition and the scalar multiplication in \mathcal{V} as follows: For all $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathcal{V}$ and all $\alpha \in \mathbb{F}$ set

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 v_1 \\ u_2 v_2 \end{bmatrix}, \quad \alpha \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} (v_1)^\alpha \\ (v_2)^\alpha \end{bmatrix}.$$

Prove that \mathcal{V} with this vector addition and this vector scaling is a vector space over \mathbb{R} .

Problem 7.3. In this problem $(-1, 1)$ denotes the open interval of real numbers. That is,

$$(-1, 1) = \{x \in \mathbb{R} : -1 < x \wedge x < 1\}.$$

For $u, v, x, y, z \in \mathbb{R}$ with $x > 0$ and $z \neq 0$ by

$$u + v, \quad u - v, \quad uv, \quad \frac{y}{z}, \quad x^u$$

we denote the standard algebraic operations in \mathbb{R} . Set $\mathcal{V} = (-1, 1)$ and let $\mathbb{F} = \mathbb{R}$. Define the vector addition and the scalar multiplication

$$\diamond : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}, \quad \diamond : \mathbb{R} \times \mathcal{V} \rightarrow \mathcal{V}$$

on \mathcal{V} by: For all $u, v \in \mathcal{V}$ and all $\alpha \in \mathbb{R}$ set

$$u \diamond v = \frac{u + v}{1 + uv}, \quad \alpha \diamond v = \frac{(1 + v)^\alpha - (1 - v)^\alpha}{(1 + v)^\alpha + (1 - v)^\alpha}.$$

Prove that \mathcal{V} with the vector addition \diamond and the scaling \diamond is a vector space over \mathbb{R} .

Problem 7.4. Consider the vector space $\mathbb{R}^{\mathbb{R}}$ of all real valued functions defined on \mathbb{R} . This vector space is considered over the field \mathbb{R} . The purpose of this exercise is to study some special subspaces of the vector space $\mathbb{R}^{\mathbb{R}}$. Let γ be an arbitrary real number. Consider the set

$$\mathcal{S}_\gamma := \left\{ f \in \mathbb{R}^{\mathbb{R}} : \exists a, b \in \mathbb{R} \text{ such that } f(t) = a \sin(\gamma t + b), t \in \mathbb{R} \right\}.$$

- Do you see exceptional values for γ for which the set \mathcal{S}_γ is particularly simple? State them and explain why they are special.
- Prove that for every $\gamma \in \mathbb{R}$ the set \mathcal{S}_γ is a subspace of $\mathbb{R}^{\mathbb{R}}$.
- For each $\gamma \in \mathbb{R}$ find a basis for \mathcal{S}_γ . Plot the function $\gamma \mapsto \dim \mathcal{S}_\gamma$.

Problem 7.5. Let D be a nonempty set and \mathbb{F} a scalar field. Let \mathbb{F}^D be a vector space introduced in Example 3.3. Let $\varphi : D \rightarrow D$ be a bijection. Set

$$\begin{aligned}\mathcal{E}_\varphi &= \{f \in \mathbb{F}^D : f(\varphi(t)) = f(t) \forall t \in D\}, \\ \mathcal{O}_\varphi &= \{f \in \mathbb{F}^D : f(\varphi(t)) = -f(t) \forall t \in D\}.\end{aligned}$$

- (a) Prove that \mathcal{E}_φ and \mathcal{O}_φ are subspaces of \mathbb{F}^D .
- (b) Prove $\mathcal{E}_\varphi \cap \mathcal{O}_\varphi = \{0_{\mathbb{F}^D}\}$.
- (c) Explore the extreme cases. Characterize the bijections $\varphi : D \rightarrow D$ such that:
 - (i) $\mathcal{E}_\varphi = \{0_{\mathbb{F}^D}\}$.
 - (ii) $\mathcal{O}_\varphi = \{0_{\mathbb{F}^D}\}$.
 - (iii) $\mathcal{E}_\varphi = \mathbb{F}^D$.
 - (iv) $\mathcal{O}_\varphi = \mathbb{F}^D$.
- (d) Explore if there is some relationship between the following three pairs of subspaces
 - (i) $\mathcal{E}_\varphi, \mathcal{O}_\varphi$.
 - (ii) $\mathcal{E}_{\varphi^{-1}}, \mathcal{O}_{\varphi^{-1}}$. (Here $\varphi^{-1} : D \rightarrow D$ is the inverse bijection of $\varphi : D \rightarrow D$.)
 - (iii) $\mathcal{E}_{\varphi \circ \varphi}, \mathcal{O}_{\varphi \circ \varphi}$.
- (e) Characterize the functions in the set $\mathcal{O}_\varphi \oplus \mathcal{E}_\varphi$.
- (f) Find a necessary and sufficient condition on $\varphi : D \rightarrow D$ for the equality $\mathbb{F}^D = \mathcal{O}_\varphi \oplus \mathcal{E}_\varphi$ to hold.

Note: This problem is inspired by the concepts of odd and even functions encountered in a precalculus class. In this precalculus setting $D = \mathbb{R}$, $\mathbb{F} = \mathbb{R}$ and $\varphi(t) = -t$ for all $t \in \mathbb{R}$. It is helpful to work out this problem for this particular case first.

Problem 7.6. Let \mathcal{V} be a vector space over a scalar field \mathbb{F} . Let \mathcal{U} and \mathcal{W} be subspaces of \mathcal{V} . Prove that $\mathcal{U} \cup \mathcal{W}$ is a subspace of \mathcal{V} if and only if $\mathcal{U} \subseteq \mathcal{W}$ or $\mathcal{W} \subseteq \mathcal{U}$.

Problem 7.7. Let \mathcal{V} be a vector space over a scalar field \mathbb{F} and let $n \in \mathbb{N}$, $n > 2$. Let $\mathcal{U}_1, \dots, \mathcal{U}_n$ be subspaces of \mathcal{V} . If the union $\mathcal{U}_1 \cup \dots \cup \mathcal{U}_n$ is a subspace, then

$$\mathcal{U}_1 \subseteq \mathcal{U}_2 \cup \dots \cup \mathcal{U}_n \quad \text{or} \quad \mathcal{U}_n \subseteq \mathcal{U}_1 \cup \dots \cup \mathcal{U}_{n-1}. \quad (7.1)$$

Proof. We will prove the contrapositive. Assume that (7.1) is not true. Then there exist $u_1 \in \mathcal{U}_1$ such that $u_1 \notin \mathcal{U}_j$ for all $j \in \{2, \dots, n\}$ and there exist $u_n \in \mathcal{U}_n$ such that $u_n \notin \mathcal{U}_j$ for all $j \in \{1, \dots, n-1\}$.

Let $\alpha \in \mathbb{F} \setminus \{0\}$. Then $\alpha u_n \in \mathcal{U}_n$ since \mathcal{U}_n is a subspace and, since $\alpha \neq 0$, $\alpha u_n \notin \mathcal{U}_j$ for all $j \in \{1, \dots, n-1\}$.

Since $u_1 \in \mathcal{U}_1$ and $\alpha u_n \notin \mathcal{U}_1$ we have $u_1 + \alpha u_n \notin \mathcal{U}_1$ for all $\alpha \in \mathbb{F} \setminus \{0\}$.

Since $u_1 \notin \mathcal{U}_n$ and $\alpha u_n \in \mathcal{U}_n$ we have $u_1 + \alpha u_n \notin \mathcal{U}_n$ for all $\alpha \in \mathbb{F}$.

Let $m \in \mathbb{N}$ be such that $1 < m < n$. (Since $n > 2$ such m exists.) By the choice of u_1 and u_n we have $u_1 \notin \mathcal{U}_m$ and $\alpha u_n \notin \mathcal{U}_m$ for all $\alpha \in \mathbb{F} \setminus \{0\}$.

Therefore, for at most one $\alpha \in \mathbb{F} \setminus \{0\}$ we can have $u_1 + \alpha u_n \in \mathcal{U}_m$. (If $u_1 + \alpha u_n \in \mathcal{U}_m$ and $u_1 + \beta u_n \in \mathcal{U}_m$ with $\alpha - \beta \neq 0$, then $(u_1 + \alpha u_n) - (u_1 + \beta u_n) = (\alpha - \beta)u_n \in \mathcal{U}_m$ with $\alpha - \beta \neq 0$ and $u_n \notin \mathcal{U}_m$ which is a contradiction.)

Thus, for at most $n - 2$ numbers $\alpha \in \mathbb{F} \setminus \{0\}$ we have

$$u_1 + \alpha u_n \in \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n.$$

Since the set $\mathbb{F} \setminus \{0\}$ is infinite, there exists $\alpha \in \mathbb{F} \setminus \{0\}$ such that

$$u_1 + \alpha u_n \notin \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n.$$

Recall that

$$u_1, u_n \in \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n.$$

The last two displayed relations show that $\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n$ is not a subspace of \mathcal{V} . \square

Problem 7.8. Let \mathcal{V} be a vector space over a scalar field \mathbb{F} and let $n \in \mathbb{N}$. Let $\mathcal{U}_1, \dots, \mathcal{U}_n$ be subspaces of \mathcal{V} . Prove that the union $\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n$ is a subspace if and only if there exists $m \in \{1, \dots, n\}$ such that $\mathcal{U}_k \subseteq \mathcal{U}_m$ for all $k \in \{1, \dots, n\}$.

Problem 7.9 (Samantha Smith). Let \mathcal{V} be a vector space over a scalar field \mathbb{F} . Let $\mathcal{P}(\mathcal{V})$ be the power set of \mathcal{V} , that is the set of all subsets of \mathcal{V} . Set $\mathcal{W} = \mathcal{P}(\mathcal{V}) \setminus \{\emptyset\}$. Let the addition and scaling in \mathcal{W} be defined as in Section 4. Is \mathcal{W} with these two operations a vector space over \mathbb{F} ?

Problem 7.10. Let \mathcal{V} be a real vector space. Set

$$\mathcal{V}_{\mathbb{C}} = \mathcal{V} \times \mathcal{V}.$$

Define the vector addition in $\mathcal{V}_{\mathbb{C}}$ as follows: For all $(u_1, v_1), (u_2, v_2) \in \mathcal{V}_{\mathbb{C}}$ set

$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2).$$

Define the vector scaling with complex numbers in $\mathcal{V}_{\mathbb{C}}$ as follows: For all $(u, v) \in \mathcal{V}_{\mathbb{C}}$ and all $\alpha, \beta \in \mathbb{R}$ set

$$(\alpha + i\beta)(u, v) = (\alpha u + \beta v, \alpha v + \beta u).$$

- Prove that $\mathcal{V}_{\mathbb{C}}$ with the vector addition and the vector scaling with complex numbers defined as above is a complex vector space.
- Prove that function

$$\iota : \mathcal{V} \rightarrow \mathcal{V}_{\mathbb{C}}$$

defined by

$$\forall v \in \mathcal{V} \quad \iota(v) = (v, 0_{\mathcal{V}})$$

is an injection which has the following property:

$$\forall \alpha, \beta \in \mathbb{R} \quad \forall u, v \in \mathcal{V} \quad \iota(\alpha u + \beta v) = \alpha \iota(u) + \beta \iota(v).$$

The range of ι is the following subset of $\mathcal{V}_{\mathbb{C}}$:

$$\{(v, 0_{\mathcal{V}}) \in \mathcal{V}_{\mathbb{C}} : v \in \mathcal{V}\}$$

The mapping ι is called the *natural embedding* of \mathcal{V} into $\mathcal{V}_{\mathbb{C}}$.

(c) Prove that for all $u, v \in \mathcal{V}$ we have

$$(u, v) = (u, 0_{\mathcal{V}}) + i(v, 0_{\mathcal{V}}).$$

Comments.

(i) the complex vector space $\mathcal{V}_{\mathbb{C}}$ is called the *complexification* of the real vector space \mathcal{V} .

(ii) Based on item (b), it is common to identify the subset

$$\{(v, 0_{\mathcal{V}}) \in \mathcal{V}_{\mathbb{C}} : v \in \mathcal{V}\}$$

with the set \mathcal{V} . Then, based on item (c) we can write

$$\mathcal{V}_{\mathbb{C}} = \mathcal{V} + i\mathcal{V}.$$

(iii) Let $n \in \mathbb{N}$. Applying the definition of $\mathcal{V}_{\mathbb{C}}$ to the real vector space \mathbb{R}^n and using the observation in the preceding item we obtain that $\mathcal{V}_{\mathbb{C}} = \mathbb{C}^n$; that is

$$(\mathbb{R}^n)_{\mathbb{C}} = \mathbb{C}^n;$$

or in words: The complexification of the real vector space \mathbb{R}^n is the complex vector space \mathbb{C}^n . The magic of this problem is that we have a universal construction which provides an embedding of an arbitrary real vector space \mathcal{V} into naturally defined complex vector space.