ON THE MAXIMUM OF A CONTINUOUS FUNCTION

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In this note a and b are real numbers such that $a < b$ and $f : [a, b] \to \mathbb{R}$ is a function.

Definition 1. If $c \in [a, b]$ and $f(c) \geq f(x)$ for all $x \in [a, b]$, then the value $f(c)$ is called a *maximum of* f.

Definition 2. A function $f : [a, b] \to \mathbb{R}$ is *continuous at a point* $x_0 \in [a, b]$ if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon, x_0) > 0$ such that

$$x \in [a, b] \ \text{ and } \ |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \epsilon.$$

The function f is *continuous on* $[a, b]$ if it is continuous at each point $x_0 \in [a, b]$.

Definition 3. Let $s \in [a, b)$. We say that a function $f : [a, b] \to \mathbb{R}$ is *dominated on* $[a, s]$ if there exists $z_0 \in (s, b]$ such that $f(x) < f(z_0)$ for all $x \in [a, s]$.

Notice the negation of this definition: f is not dominated on $[a, s]$ if for every $z \in (s, b]$ there exists $x \in [a, s]$ such that $f(z) \leq f(x)$.

A useful property of domination is:

Fact D. Let $s, t \in [a, b)$ be such that $s \leq t$. If f is dominated on $[a, t]$, then f is dominated on $[a, s]$.

Observe the contrapositive of Fact D: If f is not dominated on $[a, s]$, then f is not dominated on $[a, t]$.

Theorem. *Let* $a, b \in \mathbb{R}$, $a < b$. *If* $f : [a, b] \to \mathbb{R}$ *be a continuous function on* $[a, b]$, *then there exists* $c \in [a, b]$ *such that* $f(c)$ *is a maximum of* f.

Proof. Case I. The value $f(a)$ is a maximum of f. In this case we can set $c = a$.

Case II. The value $f(b)$ is a maximum of f. In this case we can set $c = b$.

Case III. Neither $f(a)$ nor $f(b)$ is a maximum of f. We define two sets:

$$A = \bigl\{ x \in [a, b) : \quad f \ \text{ is dominated on } \ [a, x] \ \bigr\}.$$

and

$$B = \bigl\{ y \in [a, b] \ : \ \forall\, x \in A \ \ x \leq y \bigr\}.$$

Since neither $f(a)$ nor $f(b)$ is a maximum of f, there exists $x_0 \in (a, b)$ such that $f(a) < f(x_0)$ and $f(b) < f(x_0)$. Since f is a continuous function at a and b, with

$$\epsilon_1 = \tfrac{1}{2} \min\bigl\{ f(x_0) - a, f(x_0) - b \bigr\} > 0$$

and

$$\eta = \tfrac{1}{2} \min\bigl\{ \delta(\epsilon_1, a), \delta(\epsilon_1, b), b - a \bigr\} > 0,$$

we have

$$\forall\, x \in [a, a + \eta] \cup [b - \eta, b] \quad f(x) < f(x_0). \tag{1}$$

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The strict inequality in (1) implies that $x_0 \in (a+\eta, b-\eta)$. Further, (1) implies that f is dominated on $[a, a+\eta]$. By Fact D, $[a, a+\eta] \subseteq A$.

Since $x_0 \in [a, b-\eta)$, (1) implies that f is not dominated on $[a, b-\eta]$. By the contrapositive of Fact D, it follows that f is not dominated on $[a, t]$ for every $t \in [b-\eta, b]$. Therefore $A \subseteq [a, b-\eta)$. This inclusion implies $[b-\eta, b] \subseteq B$.

Since $[a, a+\eta] \subseteq A$ and $[b-\eta, b] \subseteq B$, the sets A and B are nonempty and the Completeness Axiom applies: there exists $c \in \mathbb{R}$ such that

$$\forall x \in A \quad \forall y \in B \qquad x \leq c \leq y. \tag{2}$$

The number c whose existence is established by the Completeness Axiom has the following three special properties:

Fact 1. $a < c < b$. (To prove this we recall that $a+\eta \in A$ and $b-\eta \in B$ and set $x = a+\eta > a$ and $y = b-\eta < b$ in (2).)

Fact 2. If $s \in [a, c)$, then f is dominated on $[a, s]$. (To prove this fact, assume $s \in [a, c)$. By (2), $s \notin B$. Therefore, there exists $x \in A$ such that $s < x$. Since f is dominated on $[a, x]$ and $s < x$, Fact D implies that f is dominated on $[a, s]$.)

Fact 3. If $t \in (c, b]$, then f is not dominated on $[a, t]$. (For the proof, notice that by (2), $t \in (c, b]$ implies $t \notin A$.)

Finally, we will use the continuity of the function f at c to prove that $f(c)$ is the maximum of f. Let $\epsilon > 0$ be arbitrary. Since by Fact 1 we have $c \in (a, b)$, the definition of continuity at c implies that there exists $\delta(\epsilon)$ such that $0 < \delta(\epsilon) \leq \frac{1}{2}\min\{b-c, c-a\}$ and

$$c - \delta(\epsilon) < x < c + \delta(\epsilon) \quad \Rightarrow \quad f(c) - \epsilon < f(x) < f(c) + \epsilon. \tag{3}$$

We set

$$s = c - \delta(\epsilon)/2 \quad \text{and} \quad t = c + \delta(\epsilon)/2$$

and use the following consequence of (3):

$$\forall\, x \in [s, t] \quad f(x) < f(c) + \epsilon. \tag{4}$$

Since $s \in [a, c)$, Fact 2 yields that f is dominated on $[a, s]$. That is, there exists $z_0 \in (s, b]$ such that

$$\forall\, x \in [a, s] \quad f(x) < f(z_0). \tag{5}$$

Now we consider two cases for z_0: the first case $z_0 \in (s, t]$ and the second case $z_0 \in (t, b]$.

In the first case, (4) yields

$$f(z_0) < f(c) + \epsilon. \tag{6}$$

Inequality (6) holds true in the second case as well. To prove it, recall that $t \in (c, b]$, so, by Fact 3, we have that f is not dominated on $[a, t]$. Consequently, since in this case $z_0 \in (t, b]$, there exists $x_0 \in [a, t]$ such that $f(z_0) \leq f(x_0)$. The last inequality and (5) imply that $x_0 \in (s, t]$. Now (4) yields $f(x_0) < f(c) + \epsilon$. Since $f(z_0) \leq f(x_0)$, this proves (6) for the second case.

Statements (5) and (6) imply

$$\forall\, x \in [a, s] \quad f(x) < f(c) + \epsilon.$$

Together with (4) the last statement yields,

$$\forall\, x \in [a, t] \quad f(x) < f(c) + \epsilon. \tag{7}$$

Since $[a,t]$ is not dominated, we have

$$\forall\, z \in (t,b] \quad \exists x \in [a,t] \quad \text{such that} \quad f(z) \leq f(x). \tag{8}$$

Statements (7) and (8) yield

$$\forall\, x \in [a,b] \quad f(x) < f(c) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary,

$$\forall\, x \in [a,b] \quad f(x) \leq f(c).$$

□